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RESEARCH ARTICLE

A mathematical model of the gravitational potential of the planet taking into account tidal deformations

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Abstract
Objectives. This paper investigates the gravitational potential of a viscoelastic planet moving in the gravitational field of a massive attracting center (star), a satellite and one or more other planets moving in Keplerian elliptical orbits relative to the attracting center. Celestial bodies other than a viscoelastic planet are modeled by material points. Within the framework of the linear model of the theory of viscoelasticity, the problem of finding the vector of elastic displacement has been resolved. Traditionally, a solid body model is used to determine the Earth's gravitational field, while tidal deformations are taken into account in the form of small corrections to the coefficients of the geopotential model. In this work, the viscoelastic ball model is used to take into account tidal effects. The relevance of the research topic is associated with high-precision forecasting of the movement of artificial satellites of the Earth, high-precision measurement of the Earth's gravitational field.
Methods. In this study the asymptotic and analytical methods developed by V.G. Vilke are used for mechanical systems containing viscoelastic elements of high rigidity, as well as methods of classical mechanics, mathematical analysis. The graphs were plotted using the *Octave* mathematical package.
Results. After resolving the quasi-static problem of elasticity theory by calculating triple integrals over a spherical area, a formula for the gravitational potential of a deformable planet was obtained. In addition, the gravitational potential of the Earth was also calculated taking into account solid-state tidal effects from the Moon, Sun, and Venus at an external point. Graphs were constructed to show the dependence of the Earth's gravitational potential on time.
Conclusions. The theoretical and numerical results established herein show that the main contribution to the gravitational potential of the Earth is made by the Moon and the Sun. The influence of other planets in the solar system is small. The value of the gravitational potential at the outer point of the Earth, taking into account tidal effects, depends both on the position of the point in the moving coordinate system and on the relative position of celestial bodies.

Keywords: gravitational potential, viscoelastic planet, tides, orbit, orbit elements, mathematical modeling

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НАУЧНАЯ СТАТЬЯ

Математическая модель гравитационного потенциала планеты с учетом приливных деформаций

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Резюме
Цели. В работе исследуется гравитационный потенциал вязкоупругой планеты, совершающей движение в гравитационном поле массивного притягивающего центра (звезды), спутника и еще одной или нескольких планет, движущихся по кеплеровским эллиптическим орбитам относительно притягивающего центра. Отличные от вязкоупругой планеты небесные тела моделируются материальными точками. В рамках линейной модели теории вязкоупругости решается задача нахождения вектора упругого смещения. Традиционно для определения гравитационного поля Земли используется модель твердого тела, а учет приливных деформаций отражается в виде малых поправок к коэффициентам модели геопотенциала. В данной работе для учета приливных эффектов используется модель вязкоупругого шара. Актуальность темы исследования связана с высокоточным прогнозированием движения искусственных спутников Земли, высокоточным измерением гравитационного поля Земли.
Методы. Используются асимптотические и аналитические методы, разработанные В.Г. Вильке для механических систем, содержащих вязкоупругие элементы большой жесткости, методы классической механики, математического анализа. Построение графиков выполнено с помощью математического пакета *Octave*.
Результаты. На основе решения квазистатической задачи теории упругости путем вычисления тройных интегралов по шаровой области получена формула для гравитационного потенциала деформируемой планеты, а также вычислен гравитационный потенциал Земли с учетом твердотельных приливных эффектов от Луны, Солнца и Венеры во внешней точке. Построены графики, показывающие зависимость гравитационного потенциала Земли от времени.
Выводы. Из полученных теоретических и численных результатов следует, что основной вклад в гравитационный потенциал Земли вносят Луна и Солнце. Влияние других планет Солнечной системы мало. Значение гравитационного потенциала во внешней точке Земли с учетом приливных эффектов зависит как от положения точки в подвижной системе координат, так и от взаимного расположения небесных тел.

Ключевые слова: гравитационный потенциал, вязкоупругая планета, приливы, орбита, элементы орбиты, математическое моделирование

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INTRODUCTION

One of the most important objectives at the present time is to determine the shape of the Earth's geoid (equipotential surface of the Earth's gravity field). This is required for the topographic exploration in various scales, determination of coordinates of turning points of land plots during cadastral registration, engineering exploration, geodetic support of construction, among other reasons. In order to define the exact surface of a geoid at any point of the planet, a set of measurements need to be performed to establish the surface of the geoid or at a specific point on the Earth's surface, taking into account the nature of mass distribution. However, this is practically unfeasible.

In order to solve this task, the GRACE (a joint satellite mission of NASA and the German Aerospace Center, 2002) and GOCE (European Space Agency, 2009) spacecraft were sent into Earth orbit. These satellites have provided important data for gravimetry and higher geodesy. In particular, an improved geoid model has been constructed, which is superior in accuracy to all previous ones. Nevertheless, taking into account the so-called tidal effects, including elastic deformations of the entire globe under the influence of the gravitational force of the Moon and the Sun, all the results obtained for measuring the shape of the Earth's geoid present an average picture and are to a certain extent inaccurate.

In order to resolve the fundamental problem of determining the actual form of the Earth's geoid, considering all tidal effects, the Soviet scientist M.S. Molodensky proposed in 1950 [1, 2] that a quasigeoid should be used instead of the geoid. This is a surface close to it and does not require knowledge of the internal structure of the Earth's crust. When comparing quasigeoid and geoid, it has to be taken into account that in the case of high mountains the divergence will be approximately 2–4 m. In the case of lowland plains this divergence will be 0.02–0.12 m, and on the water surface there will be no divergence. The surface of the quasigeoid is defined by the values of the gravitational potential.

Article [3] presents an overview of the methods used to study the classical theory of tides. The traditional idea of the slowing down of the Earth's rotation under the action of tidal friction was expressed by I. Kant in 1755. The first fundamental studies on the influence of tides on the motion of planets and satellites belong to J.G. Darwin [4]. After the start of outer space exploration in the 1960s, as well as with the emergence of atomic time standards, interest in the tidal theory was revived [5].

At the present time, in accordance with the International Earth Rotation and Reference Systems Service (2010) [6] agreements, tidal deformations are recorded as small corrections to the geopotential model coefficients.

Tides in planets and their natural satellites play a crucial role in their dynamics. They are the cause of the transition to a 1:1 spin-orbit resonance [4, 5, 7]. It is also believed that tidal forces drive exoplanets into higher-order spin-orbit resonances with their parent stars [8].

Lunar-solar tides affect the change in the angular velocity of the Earth's rotation [9–11], weather and climate changes [12], and also influence volcanic activity [13]. The relevance of the research topic is related to high-precision forecasting of the motion of artificial satellites [14–17].

The aim of this study is to derive a formula for the gravitational potential of a planet modeled by a spherical viscoelastic body moving in the gravitational field of an attracting body, a natural satellite, and another planet. In order to achieve this, a motion separation method is used for mechanical systems containing viscoelastic bodies of large stiffness [18]. This work is a continuation and development of the results obtained in [19, 20].

1. TASK STATEMENT

In order to study the gravitational potential of a deformable planet, a mechanical system consisting of a stationary attracting center (material point O with mass m_1), a viscoelastic planet-satellite linkage, and another planet will be considered. The satellite and the second planet will be modeled as material points F and P with masses m_2 and m_3, respectively, and the planet being studied will be modeled as a homogeneous viscoelastic body with the shape of a ball of radius r_0 in the absence of deformations. The mass of the viscoelastic planet is m, and the density is ρ.

Let $OXYZ$ be an inertial coordinate system the origin of which coincides with the attracting center. Let $Dx_1x_2x_3$ be a mobile coordinate system associated with the viscoelastic planet with origin at its center of mass D; and C the barycenter of the viscoelastic planet–satellite system. The König axis systems $CX'Y'Z'$ and $DX''Y''Z''$ will also be introduced (Fig. 1).

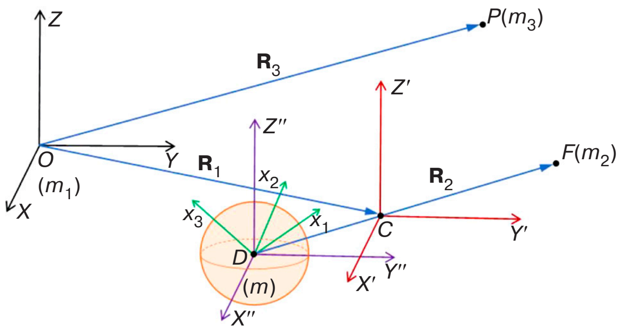

Fig. 1. Task statement

Let us assume that $\mathbf{R}_1 = \overline{OC}$ and $\mathbf{R}_2 = \overline{DF}$, $\mathbf{R}_3 = \overline{OP}$. According to the task statement

$$m_2 \ll m \ll m_1, \ |\mathbf{R}_2| \ll |\mathbf{R}_1|. \tag{1.1}$$

Deformations of a viscoelastic planet at the point M with the radius vector $\mathbf{r}$ are described by the elastic displacement vector $\mathbf{u}(\mathbf{r}, t)$. In the moving coordinate system $Dx_1x_2x_3$

$$\mathbf{u} = (u_1, u_2, u_3),\ \mathbf{r} = (x_1, x_2, x_3).$$

Radius vector of the point M in $OXYZ$ coordinate system has the following form:

$$\zeta(\mathbf{r},t) = \overline{OD} + \mathbf{\Gamma}\left(\mathbf{r} + \mathbf{u}(\mathbf{r},t)\right), \tag{1.2}$$

where $\mathbf{\Gamma}$ is the transition matrix from the moving coordinate system $Dx_1x_2x_3$ to the coordinate system $DX''Y''Z''$.

In order to determine the center of mass of the deformed planet (radius vector $\overline{OD}$) and the associated coordinate system $Dx_1x_2x_3$, the conditions [7] must be satisfied:

$$\overline{OD} = \frac{1}{m}\int_V \zeta(\mathbf{r},t)\rho\, dx,\ \int_V \mathbf{u} dx = 0,$$
$$\int_V \operatorname{rot}\mathbf{u}\, dx = 0,\ dx = dx_1dx_2dx_3, \tag{1.3}$$

where integration is performed over the ball of the radius r_0: $V = \{\mathbf{r} \in \mathbb{E}^3, |\mathbf{r}| \le r_0\}$, $\mathbb{E}^3$ is the Euclidean space.

Considering the conditions (1.3), the radius vectors $\overline{OD}$, $\overline{OF}$ of the points D and F are expressed through the vectors $\mathbf{R}_1$, $\mathbf{R}_2$ as follows:

$$\overline{OD} = \mathbf{R}_1 - \frac{m_2}{m+m_2}\mathbf{R}_2,\ \overline{OF} = \mathbf{R}_1 + \frac{m}{m+m_2}\mathbf{R}_2. \tag{1.4}$$

The potential energy of gravitational fields of the given mechanical system is represented in the following form:

$$\Pi = -\frac{fm_1m_3}{|\overline{OP}|} - \frac{fm_1m_2}{|\overline{OF}|} - \frac{fm_2m_3}{|\overline{FP}|} - \int_V \frac{fm_1\rho dx}{|\zeta(\mathbf{r},t)|} - \int_V \frac{fm_2\rho dx}{|\zeta(\mathbf{r},t) - \overline{OF}|} - \int_V \frac{fm_3\rho dx}{|\zeta(\mathbf{r},t) - \overline{OP}|},$$

where f is the universal gravitational constant.

The viscoelastic properties of the planet are described by the following parameters: Young's modulus E, Poisson's ratio ν and viscous friction coefficient χ ($\chi > 0$). Potential energy of the elastic deformations $\mathcal{E}$ and dissipative functional D are given according to the linear model [7, 18]:

$$\mathcal{E} = \int_V \mathcal{E}[\mathbf{u}]dx,\ \mathcal{E}[\mathbf{u}] = \alpha_1\left(\mathrm{I}_E^2 - \alpha_2 \mathrm{II}_E\right),$$
$$\alpha_1 = \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)},\ \alpha_2 = \frac{2(1-2\nu)}{1-\nu},$$
$$\alpha_1 > 0,\ 0 < \alpha_2 < 3,$$
$$\mathrm{I}_E = \sum_{i=1}^{3} e_{ii},\ \mathrm{II}_E = \sum_{i<j}\left(e_{ii}e_{jj} - e_{ij}^2\right), \tag{1.5}$$
$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$
$$D = \int_V D[\dot{\mathbf{u}}]dx,\ D[\dot{\mathbf{u}}] = \chi\mathcal{E}[\dot{\mathbf{u}}].$$

We will assume that the points D, C, F, P move in the plane OXY, and $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ are the given vector functions of time. In addition, we assume that the angular velocity vector $\boldsymbol{\omega}$ of the coordinate system $Dx_1x_2x_3$ is constant relatively to $DX''Y''Z''$. This vector is defined by the equation

$$\boldsymbol{\omega} \times \mathbf{a} = \mathbf{\Gamma}^{-1}\dot{\mathbf{\Gamma}}\mathbf{a},$$

where $\mathbf{a}$ is a random vector.

2. ELASTIC DISPLACEMENT VECTOR

According to the D'Alembert–Lagrange variational principle, there exists an equality [18, 21]:

$$\int_V (\ddot{\zeta}, \delta\zeta)\rho dx + \delta\Pi + \int_V \left(\nabla\mathcal{E}[\mathbf{u}] + \nabla D[\dot{\mathbf{u}}] + \boldsymbol{\lambda}_1, \delta\mathbf{u}\right)dx +$$
$$+ \boldsymbol{\lambda}_2 \int_V \operatorname{rot}\delta\mathbf{u}\, dx = 0 \qquad \forall \delta\mathbf{u} \in \left(W_2^1(V)\right)^3. \tag{2.1}$$

Lagrange multipliers $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ in (2.1) satisfy conditions (1.3); $\left(W_2^1(V)\right)^3$ is the Sobolev space.

According to (1.2) and (1.4)

$$\zeta = \mathbf{R}_1 - \frac{m_2}{m+m_2}\mathbf{R}_2 + \mathbf{\Gamma}(\mathbf{r} + \mathbf{u}).$$

Then

$$\dot{\zeta} = \dot{\mathbf{R}}_1 - \frac{m_2}{m+m_2}\dot{\mathbf{R}}_2 + \mathbf{\Gamma}\left[\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{u}) + \dot{\mathbf{u}}\right],$$

$$\ddot{\zeta} = \ddot{\mathbf{R}}_1 - \frac{m_2}{m+m_2}\ddot{\mathbf{R}}_2 +$$
$$+\ \boldsymbol{\Gamma}\left\{\boldsymbol{\omega}\times\left[\boldsymbol{\omega}\times\left(\mathbf{r}+\mathbf{u}\right)\right]+\dot{\boldsymbol{\omega}}\times\left(\mathbf{r}+\mathbf{u}\right)+2\boldsymbol{\omega}\times\dot{\mathbf{u}}+\ddot{\mathbf{u}}\right\},$$

$$\delta\zeta = \boldsymbol{\Gamma}\delta\mathbf{u}. \qquad (2.2)$$

Equality (2.2) assumes that the orbital and rotational motions of the mechanical system are specified, so the variations $\mathbf{R}_1$, $\mathbf{R}_2$, $\boldsymbol{\Gamma}$ are equal to zero.

Further:

$$\delta\Pi = \int\limits_V \left(\frac{fm_1\zeta}{|\zeta|^3}, \delta\zeta\right)\rho dx + \int\limits_V \left(\frac{fm_2\left(\zeta - \overline{OF}\right)}{\left|\zeta - \overline{OF}\right|^3}, \delta\zeta\right)\rho dx +$$
$$+\int\limits_V \left(\frac{fm_3\left(\zeta - \overline{OP}\right)}{\left|\zeta - \overline{OP}\right|^3}, \delta\zeta\right)\rho dx. \qquad (2.3)$$

Here

$$\zeta - \overline{OF} = -\mathbf{R}_2 + \boldsymbol{\Gamma}\left(\mathbf{r}+\mathbf{u}\right),$$

$$\zeta - \overline{OP} = \mathbf{R}_1 - \frac{m_2}{m+m_2}\mathbf{R}_2 - \mathbf{R}_3 + \boldsymbol{\Gamma}\left(\mathbf{r}+\mathbf{u}\right).$$

Explanation for finding the variation (2.3) of the functional Π: let $\mathbf{a} = (a_1, a_2, a_3)$, then

$$\delta\frac{1}{|\mathbf{a}|} = \delta\frac{1}{\sqrt{a_1^2+a_2^2+a_3^2}} = -\frac{1}{2}\left(a_1^2+a_2^2+a_3^2\right)^{-3/2} \times$$
$$\times\ \delta\left(a_1^2+a_2^2+a_3^2\right) = -\frac{\delta\left(a_1^2+a_2^2+a_3^2\right)}{2\left(\sqrt{a_1^2+a_2^2+a_3^2}\right)^3} =$$
$$= -\frac{2a_1\delta a_1 + 2a_2\delta a_2 + 2a_3\delta a_3}{2|\mathbf{a}|^3} = -\frac{(\mathbf{a},\delta\mathbf{a})}{|\mathbf{a}|^3}.$$

Considering (2.2):

$$\delta\Pi = \int\limits_V \left(\frac{\boldsymbol{\Gamma}^{-1}\zeta}{|\zeta|^3}, \delta\mathbf{u}\right) fm_1\rho dx +$$
$$+\int\limits_V \left(\frac{\boldsymbol{\Gamma}^{-1}\left(\zeta - \overline{OF}\right)}{\left|\zeta - \overline{OF}\right|^3}, \delta\mathbf{u}\right) fm_2\rho dx + \qquad (2.4)$$
$$+\int\limits_V \left(\frac{\boldsymbol{\Gamma}^{-1}\left(\zeta - \overline{OP}\right)}{\left|\zeta - \overline{OP}\right|^3}, \delta\mathbf{u}\right) fm_3\rho dx,$$

$$\int\limits_V \left(\ddot{\zeta}, \delta\zeta\right)\rho\, dx = \int\limits_V \left(\boldsymbol{\Gamma}^{-1}\ddot{\zeta}, \delta\mathbf{u}\right)\rho dx. \qquad (2.5)$$

We can assume that the material stiffness of the deformable planet is high, i.e., $\hat{\varepsilon} = |\boldsymbol{\omega}|^2\rho r_0^2 E^{-1} << 1$. Vectors $\mathbf{u}(\mathbf{r}, t)$, $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ can be retrieved as degree expansions $\varepsilon = E^{-1}$ [18]:

$$\mathbf{u}(\mathbf{r}, t) = \varepsilon\mathbf{u}_1(\mathbf{r}, t) + \varepsilon^2\mathbf{u}_2(\mathbf{r}, t) + \ldots,$$

$$\boldsymbol{\lambda}_1(t) = \boldsymbol{\lambda}_{10}(t) + \varepsilon\boldsymbol{\lambda}_{11}(t) + \ldots,\ \boldsymbol{\lambda}_2(t) = \boldsymbol{\lambda}_{20}(t) + \varepsilon\boldsymbol{\lambda}_{21}(t) + \ldots.$$

From Eq. (2.1), assuming (2.4), (2.5) for the vector function $\mathbf{u}_1$, we obtain the following equation:

$$\int\limits_V \left(\boldsymbol{\Gamma}^{-1}\left(\ddot{\mathbf{R}}_1 - \frac{m_2}{m+m_2}\ddot{\mathbf{R}}_2\right) + \boldsymbol{\omega}\times\left[\boldsymbol{\omega}\times\mathbf{r}\right] +\right.$$
$$+\frac{\left(\boldsymbol{\Gamma}^{-1}\overline{OD}+\mathbf{r}\right)}{\left|\boldsymbol{\Gamma}^{-1}\overline{OD}+\mathbf{r}\right|^3} fm_1 + \frac{\left(\boldsymbol{\Gamma}^{-1}\overline{FD}+\mathbf{r}\right)}{\left|\boldsymbol{\Gamma}^{-1}\overline{FD}+\mathbf{r}\right|^3} fm_2 +$$
$$\left.+\frac{\left(\boldsymbol{\Gamma}^{-1}\overline{PD}+\mathbf{r}\right)}{\left|\boldsymbol{\Gamma}^{-1}\overline{PD}+\mathbf{r}\right|^3} fm_3, \delta\mathbf{u}\right)\rho dx + \qquad (2.6)$$
$$+\int\limits_V \varepsilon\left(\nabla\mathcal{E}\left[\mathbf{u}_1\right] + \nabla D\left[\dot{\mathbf{u}}_1\right] + \boldsymbol{\lambda}_{10}, \delta\mathbf{u}\right)dx +$$
$$+\int\limits_{\partial V}\left(\boldsymbol{\lambda}_{20}\times\mathbf{n}\right)\delta\mathbf{u}\, d\sigma = 0.$$

The last summand in (2.6) is derived from the Ostrogradsky–Gauss formula, as represented below:

$$\int\limits_V \left(\boldsymbol{\lambda}_{20}, \mathrm{rot}\delta\mathbf{u}\right)dx = \int\limits_{\partial V}\left(\delta\mathbf{u}\times\boldsymbol{\lambda}_{20}\right)\mathbf{n}\, d\sigma,$$

where ∂V is the boundary of area V, and $\mathbf{n}$ is the normal to ∂V.

Following [7, 18], we substitute into (2.6) sequentially $\delta\mathbf{u} = \delta\boldsymbol{\alpha}\times\mathbf{r}$ and $\delta\mathbf{u} = \boldsymbol{\alpha}$, $\boldsymbol{\alpha}\in\mathbb{E}^3$. As a result, we obtain the following:

$$\int\limits_{\partial V}\left(\left(\boldsymbol{\lambda}_{20}\times\mathbf{n}\right),\left(\delta\boldsymbol{\alpha}\times\mathbf{r}\right)\right)d\sigma = \frac{8}{3}\pi r_0^3\left(\boldsymbol{\lambda}_{20}, \delta\boldsymbol{\alpha}\right) = 0$$

$$\forall\delta\boldsymbol{\alpha}\in\mathbb{E}^3 \Rightarrow \boldsymbol{\lambda}_{20} = 0,$$

$$\boldsymbol{\Gamma}^{-1}\left\{m\ddot{\mathbf{R}}_1 - \frac{mm_2}{m+m_2}\ddot{\mathbf{R}}_2 + \frac{fmm_1\overline{OD}}{\left|\overline{OD}\right|^3} +\right.$$
$$\left.+\frac{fmm_2\overline{FD}}{\left|\overline{FD}\right|^3} + \frac{fmm_3\overline{PD}}{\left|\overline{PD}\right|^3}\right\} + \frac{4}{3}\pi r_0^3\boldsymbol{\lambda}_{10} = 0. \qquad (2.7)$$

Also at the derivation of relation (2.7) the following equation was taken into account:

$$\int_V \frac{\mathbf{a}+\mathbf{r}}{|\mathbf{a}+\mathbf{r}|^3}\rho dx = \frac{m\mathbf{a}}{|\mathbf{a}|^3},$$

here vector $\mathbf{a}$ is independent of $\mathbf{r}$.

Since the sizes of the deformed planet are much smaller than the distances between the mutually gravitating bodies, then

$$|\mathbf{r}| \ll |\overline{OD}|, \quad |\mathbf{r}| \ll |\overline{FD}|, \quad |\mathbf{r}| \ll |\overline{PD}|.$$

If $|\mathbf{r}| << |\mathbf{a}|$ then

$$\frac{\mathbf{a}+\mathbf{r}}{|\mathbf{a}+\mathbf{r}|^3} = \frac{\mathbf{a}+\mathbf{r}}{(\mathbf{a}+\mathbf{r},\mathbf{a}+\mathbf{r})^{3/2}} = \frac{\mathbf{a}+\mathbf{r}}{\left((\mathbf{a},\mathbf{a})+2(\mathbf{a},\mathbf{r})+(\mathbf{r},\mathbf{r})\right)^{3/2}} =$$

$$= \frac{\mathbf{a}+\mathbf{r}}{|\mathbf{a}|^3}\left\{1+\frac{2(\mathbf{a},\mathbf{r})}{|\mathbf{a}|^2}+\frac{(\mathbf{r},\mathbf{r})}{|\mathbf{a}|^2}\right\}^{-3/2} \approx$$

$$\approx \frac{\mathbf{a}+\mathbf{r}}{|\mathbf{a}|^3}\left\{1-\frac{3}{2}\cdot 2\frac{(\mathbf{a},\mathbf{r})}{|\mathbf{a}|^2}\right\} = \frac{\mathbf{a}}{|\mathbf{a}|^3} - \frac{3\mathbf{a}(\mathbf{a},\mathbf{r})}{|\mathbf{a}|^5} + \frac{\mathbf{r}}{|\mathbf{a}|^3}.$$

Then Eq. (2.6) will take the form:

$$\int_V \left(\mathbf{\Gamma}^{-1}\left(\ddot{\mathbf{R}}_1 - \frac{m_2}{m+m_2}\ddot{\mathbf{R}}_2\right) + \boldsymbol{\omega}\times[\boldsymbol{\omega}\times\mathbf{r}] + \right.$$
$$+ \sum_{j=1}^{3}\left\{\frac{fm_j\mathbf{q}_j}{|\mathbf{q}_j|^3} + \frac{fm_j}{|\mathbf{q}_j|^3}\cdot\left[\mathbf{r}-3(\xi_j,\mathbf{r})\xi_j\right]\right\} + \tag{2.8}$$
$$\left. +\frac{1}{\rho}\boldsymbol{\lambda}_{10}(t),\delta\mathbf{u}\right)\rho dx + \int_V \varepsilon\left(\nabla\mathcal{E}[\mathbf{u}_1+\chi\dot{\mathbf{u}}_1],\delta\mathbf{u}\right)dx = 0.$$

Here

$$\mathbf{q}_1 = \mathbf{\Gamma}^{-1}\overline{OD} = \mathbf{\Gamma}^{-1}\left(\mathbf{R}_1 - \frac{m_2}{m+m_2}\mathbf{R}_2\right),$$
$$\mathbf{q}_2 = \mathbf{\Gamma}^{-1}\overline{FD} = -\mathbf{\Gamma}^{-1}\mathbf{R}_2,$$
$$\mathbf{q}_3 = \mathbf{\Gamma}^{-1}\overline{PD} = \mathbf{\Gamma}^{-1}\left(\mathbf{R}_1 - \frac{m_2}{m+m_2}\mathbf{R}_2 - \mathbf{R}_3\right), \tag{2.9}$$
$$\xi_i = \frac{\mathbf{q}_i}{|\mathbf{q}_i|}, \quad i = 1,\ 2,\ 3.$$

Since $m = \frac{4}{3}\pi r_0^3\rho$, the following is derived from (2.7)

$$\boldsymbol{\lambda}_{10} = -\rho\left\{\mathbf{\Gamma}^{-1}\left(\ddot{\mathbf{R}}_1 - \frac{m_2}{m+m_2}\ddot{\mathbf{R}}_2\right) + \right.$$
$$\left. + \frac{fm_1\mathbf{q}_1}{|\mathbf{q}_1|^3} + \frac{fm_2\mathbf{q}_2}{|\mathbf{q}_2|^3} + \frac{fm_3\mathbf{q}_3}{|\mathbf{q}_3|^3}\right\}. \tag{2.10}$$

Assuming (2.10), we obtain from (2.8) the following task for determining $\mathbf{u}_1(\mathbf{r}, t)$:

$$\rho\left\{\boldsymbol{\omega}\times[\boldsymbol{\omega}\times\mathbf{r}] + \sum_{j=1}^{3}\frac{fm_j}{|\mathbf{q}_j|^3}\cdot\left[\mathbf{r}-3(\xi_j,\mathbf{r})\xi_j\right]\right\} + \tag{2.11}$$
$$+ \varepsilon\nabla\mathcal{E}[\mathbf{u}_1+\chi\dot{\mathbf{u}}_1] = 0,$$

$$\boldsymbol{\sigma}_n\Big|_{r=r_0} = 0. \tag{2.12}$$

Here

$$\varepsilon\nabla\mathcal{E} = -\frac{1}{2(1+\nu)}\left(\frac{1}{1-2\nu}\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \Delta\mathbf{u}\right),$$

$$\mathbf{u} = (u_1,u_2,u_3),\ \mathbf{n} = (\gamma_1,\gamma_2,\gamma_3),\ \boldsymbol{\sigma}_n = (\sigma_{n1},\sigma_{n2},\sigma_{n3}),$$

$$\sigma_{ni} = \frac{E\nu\gamma_i}{(1+\nu)(1-2\nu)}\mathrm{div}\mathbf{u} + \frac{E}{2(1+\nu)}\left(\frac{\partial\mathbf{u}}{\partial x_i} + \mathrm{grad}u_i,\mathbf{n}\right).$$

The solution of the task (2.11)–(2.12) has the following form [7, 22]:

$$\mathbf{u}_1(\mathbf{r}, t) = \mathbf{u}_{10}(\mathbf{r}, t) + \mathbf{u}_{11}(\mathbf{r}, t) +$$
$$+ \mathbf{u}_{12}(\mathbf{r}, t) + \mathbf{u}_{13}(\mathbf{r}, t), \tag{2.13}$$

where

$$\mathbf{u}_{10}(\mathbf{r},t) = \rho\left\{\frac{2}{3}\omega^2\left[d_1r^2 + d_2r_0^2\right]\mathbf{r} + \right.$$
$$+ b_1\left[\frac{1}{6}\omega^2r^2 - \frac{1}{2}(\boldsymbol{\omega},\mathbf{r})^2\right]\mathbf{r} + \tag{2.14}$$
$$\left. + \left[b_2r^2 + b_3r_0^2\right]\cdot\left[\frac{1}{3}\omega^2\mathbf{r} - (\boldsymbol{\omega},\mathbf{r})\boldsymbol{\omega}\right]\right\},$$
$$\omega = |\boldsymbol{\omega}|,$$

$$\mathbf{u}_{1k}(\mathbf{r},t) = -\frac{3\rho fm_k}{q_k^3}\left(1+\frac{3\chi\dot{q}_k}{q_k}\right)\left\{b_1\left[\frac{1}{6}r^2 - \frac{1}{2}(\xi_k,\mathbf{r})^2\right]\mathbf{r} + \right.$$
$$\left. + \left[b_2r^2 + b_3r_0^2\right]\cdot\left[\frac{1}{3}\mathbf{r} - (\xi_k,\mathbf{r})\xi_k\right]\right\} -$$
$$-\frac{3\chi\rho fm_k}{q_k^3}\{b_1(\dot{\xi}_k,\mathbf{r})(\xi_k,\mathbf{r})\mathbf{r} + \tag{2.15}$$
$$+ [b_2r^2 + b_3r_0^2][(\dot{\xi}_k,\mathbf{r})\xi_k + (\xi_k,\mathbf{r})\dot{\xi}_k]\},$$
$$q_k = |\mathbf{q}_k|,\ k = 1,\ 2,\ 3,$$

$$d_1 = -\frac{(1+\nu)(1-2\nu)}{10(1-\nu)},\ d_2 = \frac{(1-2\nu)(3-\nu)}{10(1-\nu)},$$
$$b_1 = \frac{2(1+\nu)}{5\nu+7},\ b_2 = -\frac{(1+\nu)(2+\nu)}{5\nu+7}, \tag{2.16}$$
$$b_3 = \frac{(1+\nu)(2\nu+3)}{5\nu+7}.$$

Elastic displacement vector $\mathbf{u}$ is related to vector function $\mathbf{u}_1$ by the following equation:

$$\mathbf{u}(\mathbf{r},t)=\frac{1}{E}\mathbf{u}_1(\mathbf{r},t). \tag{2.17}$$

In formula (2.13), the summand $\mathbf{u}_{10}$ is responsible for the flattening of the planet along the rotation axis. The functions $\mathbf{u}_{1k}$ $(k = 1, 2, 3)$ describe the tidal deformations of the planet caused by the gravitational fields of the celestial bodies O, F and P.

3. GRAVITATIONAL POTENTIAL OF A VISCOELASTIC PLANET AT AN EXTERNAL POINT

Using the elastic displacement vector (2.17), the gravitational potential of the planet can be calculated by the formula:

$$\Pi(\mathbf{R},t)=-f\int\limits_V \frac{\rho dx}{\left|\boldsymbol{\Gamma}\left(\mathbf{r}+\mathbf{u}(\mathbf{r},t)\right)-\mathbf{R}\right|}. \tag{3.1}$$

Here $\mathbf{R}=\overline{DK}$ is the radius vector of the point K in which the potential is defined, $\mathbf{r}$ is the radius vector of the volume element dx with the density ρ. Vector $\mathbf{R}$ in formula (3.1) is given in the coordinate system $DX''Y''Z''$. The integral is calculated over the ball V of the radius r_0.

It is assumed that:

$$|\mathbf{R}|>r_0,\ |\mathbf{u}(\mathbf{r},t)|\ll\left|\mathbf{r}-\boldsymbol{\Gamma}^{-1}\mathbf{R}\right|. \tag{3.2}$$

According to the constructed model, the motion of points C, F, D, P occurs in the OXY plane. The coordinates of the vectors $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ in the inertial system $OXYZ$ are represented in the following form:

$$\mathbf{R}_i=R_i\left(\cos\psi_i,\sin\psi_i,0\right),\ R_i=\frac{a_i\left(1-e_i^2\right)}{\left(1+e_i\cos\vartheta_i\right)},$$
$$\psi_i=g_i+\vartheta_i,\ i=1,\ 2,\ 3,$$

a_i is the major semi-axis, e_i is the eccentricity, g_i is the pericenter longitude, ϑ_i is the true anomaly of the orbit of the end of vector $\mathbf{R}_i$. The values a_i, e_i, g_i are constant parameters of the task, and the true anomalies are time-dependent functions:

$$\dot{\vartheta}_i=\frac{\left(1+e_i\cos\vartheta_i\right)^2}{\left(1-e_i^2\right)^{3/2}}n_i,\ n_i=\frac{2\pi}{T_i},\ i=1,\ 2,\ 3. \tag{3.3}$$

The values T_i in formula (3.3) are the corresponding periods of circulation.

The transition matrix from the moving coordinate system $Dx_1x_2x_3$ to the system $DX''Y''Z''$ can be represented as a product:

$$\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_3(\psi)\boldsymbol{\Gamma}_1(\theta)\boldsymbol{\Gamma}_3(\varphi),$$

$$\boldsymbol{\Gamma}_3(\alpha)=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\boldsymbol{\Gamma}_1(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}.$$

Here ψ, θ, φ are the Euler angles [21]. The relationship between the coordinates of the angular velocity vector $\boldsymbol{\omega}$ of the planet p, q, s in the coordinate system $Dx_1x_2x_3$ and the Euler angles is expressed by means of the following kinematic Euler equations:

$$\begin{cases}p=\dot{\psi}\sin\theta\sin\varphi+\dot{\theta}\cos\varphi,\\ q=\dot{\psi}\sin\theta\cos\varphi-\dot{\theta}\sin\varphi,\\ s=\dot{\psi}\cos\theta+\dot{\varphi}.\end{cases} \tag{3.4}$$

By directing the axis Dx_3 along the vector $\boldsymbol{\omega}$, we obtain $\boldsymbol{\omega}$ = (0, 0, ω). Therefore, the following can be derived from the (3.4) system: $\dot{\psi}=0,\ \dot{\theta}=0,\ \dot{\varphi}=\omega,$ i.e., the angles ψ, θ are constant, and φ is a linear function of time. Without restriction of generality, we can assume that ψ_0 = 0. Therefore, the matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}^{-1}$ are represented by the following equations:

$$\boldsymbol{\Gamma}=\boldsymbol{\Gamma}_1(\theta_0)\boldsymbol{\Gamma}_3(\varphi),\ \boldsymbol{\Gamma}^{-1}=\boldsymbol{\Gamma}_3(-\varphi)\boldsymbol{\Gamma}_1(-\theta_0),$$
$$\psi=\psi_0(\psi_0=0),\ \theta=\theta_0,\ \varphi=\omega t+\varphi(0).$$

Let us set the radius vector of the external point K in the moving coordinate system $Dx_1x_2x_3$, using spherical coordinates R, λ, μ ($R=\left|\overline{DK}\right|$, λ is longitude, μ is latitude):

$$\mathbf{r}_K=\boldsymbol{\Gamma}^{-1}\mathbf{R}=R\mathbf{e}_R,\ \mathbf{e}_R=(\cos\lambda\cos\mu;\ \sin\lambda\cos\mu;\ \sin\mu),$$
$$R=r_0+h,\ h>0. \tag{3.5}$$

Taking into account the conditions (3.2), the gravitational potential (3.1) linearized by the components of the vector $\mathbf{u}$ has the following form:

$$\Pi(\mathbf{R},t)=-\int\limits_V \frac{f\rho dx}{\left|\mathbf{r}-\boldsymbol{\Gamma}^{-1}\mathbf{R}\right|}+ \\ +\int\limits_V \frac{\left(\mathbf{r}-\boldsymbol{\Gamma}^{-1}\mathbf{R},\mathbf{u}\right)}{\left|\mathbf{r}-\boldsymbol{\Gamma}^{-1}\mathbf{R}\right|^3}f\rho dx. \tag{3.6}$$

Let us substitute in (3.6)

$$\mathbf{u}(\mathbf{r},t) = \frac{1}{E}\sum_{k=0}^{3}\mathbf{u}_{1k}(\mathbf{r},t),$$

where the vector function $\mathbf{u}_{1k}$ is defined by the formulas (2.14)–(2.16). Let us calculate the triple integrals in (3.6) using the values of the auxiliary integrals:

$$\int_V \frac{dx}{|\mathbf{r}-\mathbf{a}|} = \frac{V_0}{|\mathbf{a}|},\ \int_V \frac{(\mathbf{r}-\mathbf{a},\mathbf{r})\,dx}{|\mathbf{r}-\mathbf{a}|^3} = 0,\ \int_V \frac{\mathbf{r}^2(\mathbf{r}-\mathbf{a},\mathbf{r})\,dx}{|\mathbf{r}-\mathbf{a}|^3} = 0,$$

$$\int_V \frac{(\boldsymbol{\omega},\mathbf{r})(\mathbf{r}-\mathbf{a},\boldsymbol{\omega})\,dx}{|\mathbf{r}-\mathbf{a}|^3} = \frac{r_0^2 V_0}{5}\left\{\frac{(\boldsymbol{\omega},\boldsymbol{\omega})}{|\mathbf{a}|^3} - \frac{3(\boldsymbol{\omega},\mathbf{a})^2}{|\mathbf{a}|^5}\right\},$$

$$\int_V \frac{(\boldsymbol{\omega},\mathbf{r})^2(\mathbf{r}-\mathbf{a},\mathbf{r})\,dx}{|\mathbf{r}-\mathbf{a}|^3} = \frac{2r_0^4 V_0}{35}\left\{\frac{(\boldsymbol{\omega},\boldsymbol{\omega})}{|\mathbf{a}|^3} - \frac{3(\boldsymbol{\omega},\mathbf{a})^2}{|\mathbf{a}|^5}\right\},$$

$$\int_V \frac{\mathbf{r}^2(\boldsymbol{\omega},\mathbf{r})(\mathbf{r}-\mathbf{a},\boldsymbol{\omega})\,dx}{|\mathbf{r}-\mathbf{a}|^3} = \frac{r_0^4 V_0}{7}\left\{\frac{(\boldsymbol{\omega},\boldsymbol{\omega})}{|\mathbf{a}|^3} - \frac{3(\boldsymbol{\omega},\mathbf{a})^2}{|\mathbf{a}|^5}\right\},$$

where $V_0 = 4\pi r_0^3/3$. After calculating triple integrals over the spherical area V we obtain the following expression for the gravitational potential:

$$\Pi = -\frac{fm}{R} - \frac{3fm^2 r_0 \Phi(\nu)}{140\pi E R^3}\left\{\omega^2 - 3(\omega,\mathbf{e}_R)^2\right\} + \\ + \sum_{k=1}^{3}\frac{9f^2m^2m_k r_0\Phi(\nu)}{140\pi E q_k^3 R^3}\left\{1-3(\xi_k,\mathbf{e}_R)^2\right\}\left(1+\frac{3\chi\dot{q}_k}{q_k}\right) + \\ + \sum_{k=1}^{3}\frac{27\chi f^2m^2m_k r_0\Phi(\nu)}{70\pi E q_k^3 R^3}(\dot{\xi}_k,\mathbf{e}_R)(\xi_k,\mathbf{e}_R). \quad (3.7)$$

Here

$$R = |\mathbf{R}|,\ \mathbf{e}_R = \boldsymbol{\Gamma}^{-1}\frac{\mathbf{R}}{R},\ \Phi(\nu) = \frac{(1+\nu)(9\nu+13)}{(5\nu+7)},$$
$$q_k = |\mathbf{q}_k|,\ k = 1, 2, 3.$$

The singular vector $\mathbf{e}_R$ in the moving coordinate system $Dx_1x_2x_3$ is given by the spherical coordinates λ and μ according to (3.5).

Let us express q_k through the orbital elements. According to (2.9)

$$\mathbf{q}_1 = \boldsymbol{\Gamma}^{-1}\left(\mathbf{R}_1 - \frac{m_2}{m+m_2}\mathbf{R}_2\right),$$
$$q_1 = |\mathbf{q}_1| = \quad (3.8)$$
$$= R_1\sqrt{1 - \frac{2m_2}{m+m_2}\frac{R_2}{R_1}\cos\gamma_{12} + \left(\frac{m_2}{m+m_2}\right)^2\left(\frac{R_2}{R_1}\right)^2},$$

where $\gamma_{12} = g_1 + \vartheta_1 - g_2 - \vartheta_2$ is the angle between vectors $\mathbf{R}_1$ и $\mathbf{R}_2$, $R_1 = |\mathbf{R}_1|$, $R_2 = |\mathbf{R}_2|$.

Due to condition (1.1)

$$\frac{m_2}{m+m_2}\cdot\frac{R_2}{R_1} \ll 1.$$

Let us rewrite the formula (3.8) in the form:

$$q_1 = \frac{a_1(1-e_1^2)}{(1+e_1\cos\vartheta_1)}F_{21},$$

$$F_{21} = \sqrt{1 - 2k_{21}h_{21}\cos\gamma_{21} + k_{21}^2h_{21}^2},$$

$$k_{21} = \frac{m_2}{m+m_2}\frac{a_2(1-e_2^2)}{a_1(1-e_1^2)},\ h_{21} = \frac{1+e_1\cos\vartheta_1}{1+e_2\cos\vartheta_2}. \quad (3.9)$$

Further:

$$q_2 = \frac{a_2(1-e_2^2)}{(1+e_2\cos\vartheta_2)}, \quad (3.10)$$

$$q_3 = \sqrt{R_1^2 - 2(\mathbf{R}_1,\mathbf{R}_3) + R_3^2} = \frac{a_3(1-e_3^2)}{1+e_3\cos\vartheta_3}F_{13},$$

$$F_{13} = \sqrt{1 - 2k_{13}h_{13}\cos\gamma_{13} + k_{13}^2h_{13}^2},$$

$$k_{13} = \frac{a_1(1-e_1^2)}{a_3(1-e_3^2)},\ h_{13} = \frac{1+e_3\cos\vartheta_3}{1+e_1\cos\vartheta_1}. \quad (3.11)$$

Let us find the coordinates of the vectors $\boldsymbol{\Gamma}^{-1}\mathbf{R}_k$, $k = 1, 2, 3$:

$$\boldsymbol{\Gamma}^{-1}\mathbf{R}_k = \boldsymbol{\Gamma}_3(-\varphi)\boldsymbol{\Gamma}_1(-\theta_0)\mathbf{R}_k = \\ = R_k\boldsymbol{\eta}_k = \frac{a_k(1-e_k^2)}{(1+e_k\cos\vartheta_k)}\boldsymbol{\eta}_k, \quad (3.12)$$

$$\boldsymbol{\eta}_k = (\eta_{k1},\eta_{k2},\eta_{k3}), \quad (3.13)$$

$$\eta_{k1} = \cos\varphi\cos(g_k+\vartheta_k) + \sin\varphi\cos\theta_0\sin(g_k+\vartheta_k),$$

$$\eta_{k2} = -\sin\varphi\cos(g_k+\vartheta_k) + \cos\varphi\cos\theta_0\sin(g_k+\vartheta_k),$$

$$\eta_{k3} = -\sin\theta_0\sin(g_k+\vartheta_k).$$

Using (2.9) and (3.12), we find the coordinates of the vectors ξ_k:

$$\xi_1 = \frac{1}{F_{21}}(\boldsymbol{\eta}_1 - k_{21}h_{21}\boldsymbol{\eta}_2),$$
$$\xi_2 = -\boldsymbol{\eta}_2,\ \xi_3 = \frac{1}{F_{13}}(k_{13}h_{13}\boldsymbol{\eta}_1 - \boldsymbol{\eta}_3). \quad (3.14)$$

Thus, the computation of scalar products (ξ_k, $\mathbf{e}_R$) is reduced to the computation of scalar products ($\boldsymbol{\eta}_k$, $\mathbf{e}_R$). Using (3.5), (3.13), (3.14), we obtain:

$$\begin{gathered}(\boldsymbol{\eta}_k,\mathbf{e}_R)=\left[\cos(\varphi+\lambda)\cos(g_k+\vartheta_k)+\right.\\ \left.+\sin(\varphi+\lambda)\cos\theta_0\sin(g_k+\vartheta_k)\right]\cdot\cos\mu- \\ -\sin\theta_0\sin(g_k+\vartheta_k)\sin\mu.\end{gathered}\tag{3.15}$$

From (3.14) and (3.15) the following can be derived:

$$\begin{gathered}(\xi_1,\mathbf{e}_R)=\frac{1}{F_{21}}\left[(\boldsymbol{\eta}_1,\mathbf{e}_R)-k_{21}h_{21}\boldsymbol{\eta}_2\right],\\ (\xi_2,\mathbf{e}_R)=-(\boldsymbol{\eta}_2,\mathbf{e}_R),\\ (\xi_3,\mathbf{e}_R)=\frac{1}{F_{13}}\left(k_{13}h_{13}(\boldsymbol{\eta}_1,\mathbf{e}_R)-(\boldsymbol{\eta}_3,\mathbf{e}_R)\right).\end{gathered}\tag{3.16}$$

Since $\boldsymbol{\omega}=(0, 0, \omega)$, then

$$(\boldsymbol{\omega},\mathbf{e}_R)=\omega\sin\mu.\tag{3.17}$$

Taking into account that the influence of dissipative forces at time intervals commensurate with the period of the planet's revolution around its own axis is negligible, we transform formula (3.7) by putting $\chi = 0$:

$$\begin{gathered}\Pi=-\frac{fm}{R}-\frac{3fm^2r_0\Phi(\nu)}{140\pi ER^3}\left\{\omega^2-3(\boldsymbol{\omega},\mathbf{e}_R)^2\right\}+\\ +\sum_{k=1}^{3}\frac{9f^2m^2m_kr_0\Phi(\nu)}{140\pi Eq_k^3R^3}\left\{1-3(\xi_k,\mathbf{e}_R)^2\right\}.\end{gathered}\tag{3.18}$$

According to formula (3.18), in order to estimate the contribution to the Earth's gravitational potential from the Sun, the Moon and other planets of the Solar System, it is necessary to estimate the value $\dfrac{m_k}{q_k^3}$. Index $k = 1$ corresponds to the influence of the Sun. Index $k = 2$ corresponds to the influence of the Moon, and $k = 3$ corresponds to the influence of the planet. Assuming all orbits to be circular, we obtain:

$$\begin{gathered}\frac{m_1}{q_1^3}\approx\frac{m_1}{R_1^3},\quad\frac{m_2}{q_2^3}=\frac{m_2}{R_2^3},\\ \frac{m_3}{(R_1+R_2)^3}\le\frac{m_3}{q_3^3}\le\frac{m_3}{(R_1-R_2)^3}.\end{gathered}$$

For the Sun and the Moon, the values of this quantity expressed in 10^{24} kg/(a.u.)^3 are as follows:

$$m_1/q_1^3=1.989\cdot10^6,\ m_2/q_2^3=4.332\cdot10^6.$$

Table. Influence of the gravitational fields of the Solar System planets on the Earth's gravitational potential (measuring unit 10^{24} kg/(a.u.)^3)

Planets	$\dfrac{m_3}{(R_1+R_2)^3}$	$\dfrac{m_3}{(R_1-R_2)^3}$
Mercury	0.124	1.434
Venus	0.951	229.889
Mars	0.040	4.470
Jupiter	7.953	25.565
Saturn	0.486	0.914
Uranus	0.011	0.014
Neptune	$3.415\cdot10^{-3}$	$4.17\cdot10^{-3}$

The results of the calculations of the minimum and maximum values of the value m_3/q_3^3 corresponding to the planets of the solar system are presented in the table. The numerical values are expressed in 10^{24} kg/(a.u.)^3 (1 a.u. = $1.495978707\cdot10^{11}$ m). This is derived from the results obtained that the main contribution to the Earth's gravitational potential comes from the Moon and the Sun. The influence of other planets of the Solar System is not significant. The maximum value of the parameter m_k/q_k^3 is reached by Venus at the moment of its maximum approach to the Earth. However, even this value is of the order of 10^{-4}, when compared to the values of the corresponding numerical coefficients for the Moon and the Sun.

Let us transform the expression (3.18) by selecting dimensionless coefficients and taking into account (3.16)–(3.17), thus we obtain the following formula for the gravitational potential of a viscoelastic planet taking into account tidal effects:

$$\begin{gathered}\Pi=-\frac{fm}{R}\left\{1+k_0\left(1-3\sin^2\mu\right)+\frac{k_1(1+e_1\cos\vartheta_1)^3}{F_{21}^3}+\right.\\ +\left(\frac{3}{F_{21}^2}\left[(\boldsymbol{\eta}_1,\mathbf{e}_R)-k_{21}h_{21}(\boldsymbol{\eta}_2,\mathbf{e}_R)\right]^2-1\right)+\\ +k_2(1+e_2\cos\vartheta_2)^3\left(3(\boldsymbol{\eta}_2,\mathbf{e}_R)^2-1\right)+\\ +\frac{k_3(1+e_3\cos\vartheta_3)^3}{F_{13}^3}\times\\ \left.\times\left(\frac{3}{F_{13}^2}\left[k_{13}h_{13}(\boldsymbol{\eta}_1,\mathbf{e}_R)-(\boldsymbol{\eta}_3,\mathbf{e}_R)\right]^2-1\right)\right\},\end{gathered}\tag{3.19}$$

where

$$\begin{gathered}k_0=\frac{3mr_0\omega^2\Phi(\nu)}{140\pi ER^2},\\ k_j=\frac{9fmm_jr_0\Phi(\nu)}{140\pi ER^2a_j^3\left(1-e_j^2\right)^3},\ j=1,2,3.\end{gathered}\tag{3.20}$$

Let us apply the obtained result to calculate the gravitational potential of the Earth moving in the gravitational field of the Sun, Moon and Venus. Let us take the following values of the parameters included in formulas (3.3), (3.9)–(3.11), (3.20) [23]:

$$r_0 = 6.378 \cdot 10^6 \text{ m}, h = 3 \cdot 10^5 \text{ m}, E = 1.2 \cdot 10^{11} \text{ kg/(m} \cdot \text{s}^2),$$
$$\nu = 0.2, \omega = 7.2922 \cdot 10^{-5} \text{ s}^{-1},$$

$$f = 6.672 \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2), m = 5.9736 \cdot 10^{24} \text{ kg},$$
$$m_1 = 1.98911 \cdot 10^{30} \text{ kg}, m_2 = 7.349 \cdot 10^{22} \text{ kg},$$
$$m_3 = 4.8685 \cdot 10^{24} \text{ kg},$$

$$a_1 = 1.4959787 \cdot 10^{11} \text{ m}, a_2 = 3.844 \cdot 10^8 \text{ m},$$
$$a_3 = 108208627813 \text{ m},$$

$$e_1 = 0.01671022, e_2 = 0.0549, e_3 = 0.00676,$$
$$\theta_0 = 23.45° = 0.409280 \text{ rad},$$

$$T_1 = 365.26 \text{ days}, T_2 = 27.321661 \text{ days},$$
$$T_3 = 224.7 \text{ days},$$

$$g_1 = 1.7967674 \text{ rad}, g_2 = 0 \text{ rad}, g_3 = 2.2956836 \text{ rad}.$$

Dimensionless coefficients included in (3.19) have the following values:

$$k_0 = 5.73271 \cdot 10^{-4}, k_1 = 1.283118 \cdot 10^{-8},$$
$$k_2 = 2.817287 \cdot 10^{-8}, k_3 = 8.2926 \cdot 10^{-14},$$
$$k_{21} = 3.114226 \cdot 10^{-5}, k_{13} = 1.382169.$$

As dimensionless time, we take the number of revolutions of the Earth around its own axis:

$$\tau = \frac{\varphi}{2\pi} = \frac{\omega t + \varphi(0)}{2\pi}.$$

By denoting by the stroke the derivative of τ, we obtain from (3.3):

$$\vartheta_i' = N_i\left(1 + e_i \cos\vartheta_i\right)^2, \; N_i = \frac{2\pi T_{\text{rot}}}{\left(1 - e_i^2\right)^{3/2} T_i}, \; i = 1, 2, 3,$$

where T_{rot} = 23.93419 h is the period of the Earth's revolution around its own axis. Values of dimensionless coefficients N_i:

$$N_1 = 0.017162, N_2 = 0.230381, N_3 = 0.027888.$$

Let us introduce into consideration a dimensionless function—the relative gravitational potential Π_1:

$$\Pi_1 = \frac{\Pi - \Pi_0}{\Pi_0}, \; \Pi_0 = -\frac{fm}{R}.$$

According to (3.19)

$$\Pi_1 = \Pi_{10} + \Pi_{11}, \Pi_{10} = k_0(1 - 3\sin^2\mu) = \text{const},$$

$$\Pi_{11} = \frac{k_1\left(1 + e_1 \cos\vartheta_1\right)^3}{F_{21}^3} \times$$
$$\times \left(\frac{3}{F_{21}^2}\left[\left(\boldsymbol{\eta}_1, \mathbf{e}_R\right) - k_{21}h_{21}\left(\boldsymbol{\eta}_2, \mathbf{e}_R\right)\right]^2 - 1\right) +$$
$$+ k_2\left(1 + e_2 \cos\vartheta_2\right)^3\left(3\left(\boldsymbol{\eta}_2, \mathbf{e}_R\right)^2 - 1\right) + \tag{3.21}$$
$$+ \frac{k_3\left(1 + e_3 \cos\vartheta_3\right)^3}{F_{13}^3} \times$$
$$\times \left(\frac{3}{F_{13}^2}\left[k_{13}h_{13}\left(\boldsymbol{\eta}_1, \mathbf{e}_R\right) - \left(\boldsymbol{\eta}_3, \mathbf{e}_R\right)\right]^2 - 1\right).$$

The Π_{10} summand is responsible for the perturbing part of the gravitational potential caused by the Earth's compression along the rotation axis, which does not change with time. The function Π_{11}, given by formula (3.21), is the time-dependent part of the gravitational potential caused by tidal effects.

Figures 2–4 show the graphs obtained using the *Octave*[1] mathematical package of the function $\Pi_{11} = \Pi_{11}(\tau)$ describing the Earth's gravitational potential at an external point over a period of 30 days at an altitude of h = 300 km from the Earth's surface at different latitudes (0°, 30°, 60°). At the initial moment of time, the following parameters are set:

$$\vartheta_1(0) = -0.0433335, \; \vartheta_2(0) = 0, \; \vartheta_3(0) = 0.8804619,$$
$$g_1 = 1.7967674, \; g_2 = 0, \; g_3 = 2.2956836, \; \lambda = 0.$$

One graduation on the abscissa axis corresponds to two days.

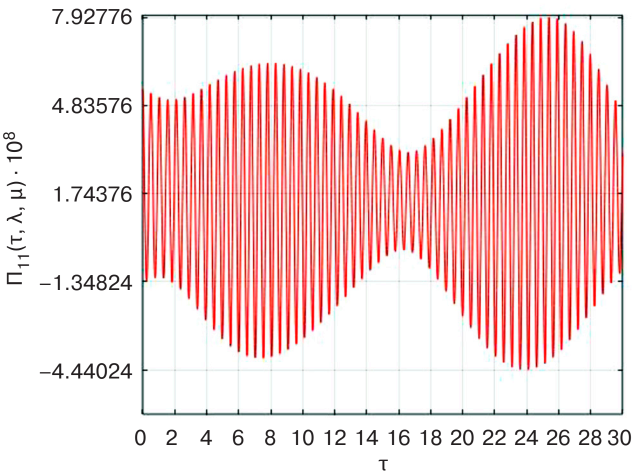

Fig. 2. Change in the Earth's gravitational potential at an external point with spherical coordinates $R = 6.678 \cdot 10^6$ m, $\lambda = 0$, $\mu = 0$ for 30 days

[1] https://octave.org/. Accessed May 16, 2023.

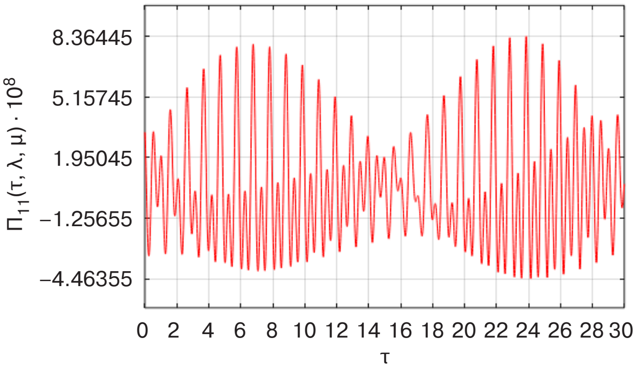

Fig. 3. Change in the Earth's gravitational potential at an external point with spherical coordinates $R = 6.678 \cdot 10^6$ m, $\lambda = 0$, $\mu = \pi/6$ for 30 days

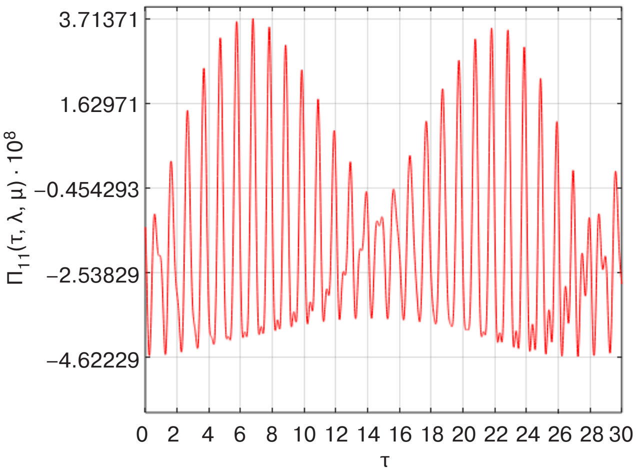

Fig. 4. Change in the Earth's gravitational potential at an external point with spherical coordinates $R = 6.678 \cdot 10^6$ m, $\lambda = 0$, $\mu = \pi/3$ for 30 days

As can be seen from the above graphs, the dependence of the gravitational potential on time has a complex oscillatory character. It depends significantly on the geographic latitude of the point over which it is measured.

CONCLUSIONS

In this article, a formula has been established for the gravitational potential of a planet modeled by a viscoelastic ball. It considers the tidal effects caused by the gravitational fields of its natural satellite, the attracting center (star) and another planet. Based on the model thus constructed, it is shown that the main contribution to the tidal component of the Earth's gravitational potential is made by the Moon and the Sun. The study also makes estimates of the influence of gravitational fields of the other planets of the Solar System on the gravitational potential of the Earth, modeled as a viscoelastic body. Graphs have been plotted of the time dependence of the tidal component of the gravitational potential at three different points in the Earth-related coordinate system located at different latitudes at an altitude of 300 km from the Earth's surface.

Authors' contribution. All authors equally contributed to the research work.

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