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RESEARCH ARTICLE

The use of complex structure splines in roadway design

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Abstract
Objectives. The aim of the work is to develop the theory of spline-approximation of a sequence of points on a plane for using compound splines with a complex structure. In contrast to a simple spline (e.g., polynomial), a compound spline contains repeating bundles of several elements. Such problems typically arise in the design of traces for railroads and highways. The plan (projection on the horizontal plane) of such a trace is a curve consisting of a repeating bundle of elements "line + clothoid + circle + clothoid ...," which ensures continuity not only of curve and tangent but also of curvature. The number of spline elements, which is unknown, should be determined in the process of solving the design problem. An algorithm for solving the problem with respect to the spline, which consists of arcs conjugated by straight lines, was implemented and published in an earlier work. The approximating spline in the general case is a multivalued function, whose ordinates may be limited. Another significant factor that complicates the problem is the presence of clothoids that are not expressed analytically (in a formula). The algorithm for determining the number of elements of a spline with clothoids and constructing an initial approximation was also published earlier. The present work considers the next stage of solving the spline approximation problem: optimization using a nonlinear programming spline obtained at the first stage by means of the dynamic programming method.
Methods. A new mathematical model in the form of a modified Lagrange function is used together with a special nonlinear programming algorithm to optimize spline parameters. In this case, it is possible to calculate the derivatives of the objective function by the spline parameters in the absence of its analytical expression through these parameters.
Results. A mathematical model and algorithm for optimization of compound spline parameters comprising arcs of circles conjugated by clothoids and lines have been developed.
Conclusions. The previously proposed two-step scheme for designing paths of linear structures is also suitable for the utilization of compound splines with clothoids.

Keywords: trace plan, spline, nonlinear programming, clothoid, objective function, constraints

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НАУЧНАЯ СТАТЬЯ

Использование сплайнов сложной структуры в проектировании дорожных трасс

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Резюме
Цели. Цель работы состоит в развитии теории сплайн-аппроксимации последовательности точек на плоскости на случай использования составных сплайнов сложной структуры. В отличие от простого, например, полиномиального сплайна, составной сплайн содержит повторяющиеся связки нескольких элементов. Такая задача возникает в проектировании трасс железных и автомобильных дорог. План (проекция на горизонтальную плоскость) такой трассы – это кривая, состоящая из повторяющейся связки элементов «прямая + + клотоида + окружность + клотоида …», что обеспечивает непрерывность не только кривой и касательной, но и кривизны. Число элементов сплайна неизвестно и должно определяться в процессе решения проектной задачи. Алгоритм решения задачи применительно к сплайну, состоящему из дуг окружностей, сопрягаемых прямыми, реализован и опубликован ранее. Аппроксимирующий сплайн в общем случае – многозначная функция. На координаты точек ее графика могут накладываться ограничения. Еще одним существенным фактором, усложняющим задачу, является наличие клотоид, которые не выражаются аналитически (формулой). Алгоритм определения числа элементов сплайна с клотоидами и построения начального приближения опубликован ранее. В настоящей статье рассматривается следующий этап решения задачи – оптимизация с применением нелинейного программирования сплайна, полученного на первом этапе по методу динамического программирования.
Методы. Для оптимизации параметров сплайна используется новая математическая модель в виде модифицированной функции Лагранжа и специальный алгоритм нелинейного программирования. При этом удается вычислять аналитически производные целевой функции по параметрам сплайна при отсутствии ее аналитического выражения через эти параметры.
Результаты. Разработаны математическая модель и алгоритм оптимизации параметров составного сплайна, состоящего из дуг окружностей, сопрягаемых клотоидами и прямыми.
Выводы. Предложенная ранее двухэтапная схема проектирования плана трасс линейных сооружений пригодна и при использовании составных сплайнов с клотоидами.

Ключевые слова: план трассы, сплайн, нелинейное программирование, клотоида, целевая функция, ограничения

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INTRODUCTION

The method of approximating a given sequence of points in the plane using a special kind of spline implies a two-stage scheme for obtaining a solution [1]. Here, the first stage consists in obtaining the number of spline elements and approximate parameter values using the dynamic programming method. At the second stage, optimization of the parameters of obtained spline used as an initial approximation is performed using nonlinear programming. The first stage was considered in [1]. In the present article, representing the culmination of a series of articles [1–3] devoted to spline approximation methods, the second stage is considered in relation to the use of a spline with clothoids for conjugating straight lines with circles. The solution to this problem with respect to a spline consisting of arcs of circles conjugated by straight lines was presented in [3]. The results of this earlier work are referred to in the present article, which develops the model and algorithm [3] for the more complex case of a spline with clothoids.

A spline consists of a repeated conjunction "line segment + clothoid arc + circle arc + clothoid arc...." In what follows, the word "arc" will be omitted for brevity unless ambiguity arises. At this stage, the initial point and the direction of the tangent in it, as well as the lengths of all curves and lines conjugating them, are known. This allows us to apply continuous optimization methods—in particular, methods of nonlinear programming of the gradient type—despite the desired spline in the general case representing a multivalued function.

The problem is considered in relation to the design of route plans for railroads and highways, where—unlike other linear structures, such as pipelines—clothoids are a necessary means of ensuring curvature continuity to ensure traffic comfort and safety.[1,2]

In this connection, the accepted approach is noted to differ significantly from the method of selecting elements in interactive mode used in design practice, as well as from various semi-automatic methods of searching curve boundaries on the basis of curvature and angle diagrams, and from the new heuristic method of searching curve boundaries [4] with the subsequent application of genetic optimization algorithms [5–14].

Consequently, the use of adequate mathematical models and mathematically correct algorithms seems to represent a more promising approach.

[1] SP 34.13330.2012. *Automobile roads*. Updated edition of SNiP 2.05.02-85* (with Amendments No. 1, 2). Code of Regulations. June 30, 2012. https://docs.cntd.ru/document/1200095524 (in Russ.). Accessed December, 20, 2023.

[2] SP 119.13330.2017. *Railway with 1520 mm track*. Updated edition of SNiP 32-01-95 (with Amendment No. 1). https://docs.cntd.ru/document/550965737 (in Russ.). Accessed December, 20, 2023.

1. TASK STATEMENT AND ITS FORMALIZATION

The task statement and its formalization do not differ significantly from that presented in [3] when solving the problem without clothoids. However, the presence of clothoids creates significant difficulties in the realization of gradient calculation concepts for the application of nonlinear programming.

A clothoid represents a plane curve (Fig. 1) whose curvature σ depends linearly on its length l. Thus, for a piece of clothoid with an arbitrary initial point A, curvature at this point σ_A, and the end point B with curvature σ_B, we have the formula:

$$\sigma_B = \sigma_A + kL, \quad (1)$$

where L is the length of the clothoid piece and k is its parameter.

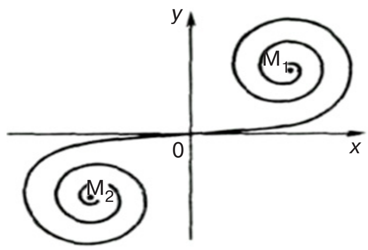

Fig. 1. Clothoid

This linear dependence is the basis for all subsequent actions in calculating derivatives in order to apply mathematical programming.

The task is as follows: to find a spline of a given form, which satisfies all constraints and best approximates a given sequence of points in the plane (Fig. 2).

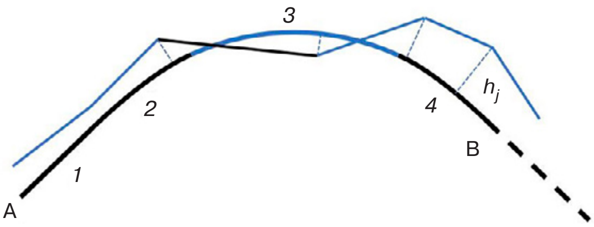

Fig. 2. One spline bundle:
1—straight line, *2* and *4*—clothoids, *3*—circle

The preset initial point A and direction of the tangent to the desired spline at this point do not change during the optimization process.

Approximation quality is estimated by the sum of squares of deviations h_j (Fig. 2) of the given points from the spline. In other words, h_j represents the displacement of a given point to its designed position, calculated along the normal to the original broken line [3], i.e., along the direction to the center of the circle connecting three adjacent points. If three points lie on the same line, h_j are calculated along the normal to this line.

Offsets of the initial points to the design position are considered to be positive if they are carried out in the direction of the external normal.

Now it is necessary to obtain

$$\min F(\mathbf{h}) = 1/2 \sum_{1}^{n} h_j^2. \quad (2)$$

Here $\mathbf{h}(h_1, h_2, \ldots, h_n)$ is the vector of variables; n is their number. A weighted sum of squares can be specified instead of a simple sum.

Since each variable is constrained separately, the system of constraints on the main variables contains simple inequalities. This system is practically the same as in [3]. Only the constraint on the length of the clothoid is added; instead of a variable radius, a variable curvature is considered. The constraints on the individual displacements hm are the same as in [3].

It is not possible to express the conditions of presence and position of lines, clothoids and circles through the variables h_j. We consider these variables as intermediate variables, while the main variables are the lengths of lines, clothoids and circles, as well as the curvatures of circles.

In formal terms, the mathematical programming problem is the same as in [3]. However, since the presence of clothoids significantly complicates the task, it requires a separate consideration.

2. TASK FEATURES

A spline is completely defined by the main variables, taking into account the initial point and the direction of the tangent in it. However, we lack analytical expressions of dependencies (formulas) of intermediate variables on the main variables. The constraints on the main variables are not expressed through the intermediate variables. Moreover, there is no analytical dependence of the objective function (2) on the main variables.

A clothoid cannot be generally represented in a Cartesian coordinate system by a function $y(x)$.

If the origin of the coordinate system coincides with the point of zero curvature of the clothoid, and the OX axis is a tangent at this point (Fig. 1), then the parametric representation of the x and y coordinates as functions of the length l counted from the point of zero curvature in the form of degree series is used in this case:

$$x(l) \approx l\left(1 - \frac{l^4k^2}{40} + \frac{l^8k^4}{3456} - \ldots\right),$$
$$y(l) \approx \frac{l^3k}{6}\left(1 - \frac{l^4k^2}{56} + \frac{l^8k^4}{7040} - \ldots\right). \quad (3)$$

For expanding the series, formulas for the clothoid in an arbitrary coordinate system are obtained, taking into account the coordinates of the initial point, the angle of the tangent in it to the OX axis, and the curvature [15].

Due to the noted features of the problem, the idea of solving it as a nonlinear programming task using gradient methods [16–18] seems unfeasible. However, the task of spline approximation using circles conjugated by straight lines was solved in this way [3] despite the lack of analytical expressions of differentiable functions. After obtaining formulas for derivatives of intermediate variables h_j on the main variables, we were able to easily calculate the derivatives of the objective function on the main variables [3].

3. INTEGRAL REPRESENTATION OF THE CLOTHOID AND ITS APPLICATION

Since, for any smooth curve, $\sigma = d\varphi/dl$, where σ is the curvature, φ and l are the respective current values of the tangent angle with the OX axis and length, we derive from (1):

$$\varphi_B = \varphi_A + \sigma_A L + kL^2/2 = \varphi_A + L(\sigma_A + \sigma_B)/2. \quad (4)$$

Between the arc length increment of any smooth curve and the coordinate increment, there are the relations $dx = \cos\varphi\, dl$ and $dy = \sin\varphi\, dl$, from which, using (4) to denote the integration variable by t, we derive a representation of the clothoid in parametric form:

$$x(l) = x_A + \int_0^l \cos(\varphi_A + \sigma_A t + kt^2/2)dt,$$
$$y(l) = y_A + \int_0^l \sin(\varphi_A + \sigma_A t + kt^2/2)dt. \quad (5)$$

Here x_A, y_A are the coordinates of the initial point and l is the length of the piece of clothoid from the initial point A to the current point with coordinates $x(l)$, $y(l)$.

Further, we will rely on the parametric representation of the clothoid (5).

Let us consider what transformations occur with the spline when changing one and only one main variable. An understanding of these transformations will be used to generate formulas for calculating partial derivatives of intermediate variables (h_j) from element lengths and curvatures, i.e., by basic variables.

When changing the length of a line by ΔL, the right part of the spline is shifted in the direction of this line. When changing the length of the circle arc, a shift in the direction of the tangent at the end point of the arc plus rotation is centered at this point by $\Delta\varphi = \sigma\Delta L$. When the length of the clothoid changes, the following occurs:

1. The clothoid parameter is varied in such a way that the curvature at the end point does not change when the length changes, since we calculate partial derivatives.
2. The coordinates of the right end (point B) of the clothoid and the angle of the tangent in it with the *OX* axis change. According to (5)

$$x_B = x_A + \int_0^L \cos(\varphi_A + \sigma_A t + kt^2/2)dt, \quad y_B = y_A + \int_0^L \sin(\varphi_A + \sigma_A t + kt^2/2)dt, \tag{6}$$

$$\frac{\partial x_B}{\partial L} = \cos\varphi_B + \frac{\partial x_B}{\partial k}\cdot\frac{\partial k}{\partial L} = = \cos\varphi_B - \frac{\partial x_B}{\partial k}\cdot\frac{(\sigma_B - \sigma_A)}{L^2}, \tag{7}$$

$$\frac{\partial y_B}{\partial L} = \sin\varphi_B + \frac{\partial y_B}{\partial k}\cdot\frac{\partial k}{\partial L} = = \sin\varphi_B - \frac{\partial y_B}{\partial k}\cdot\frac{(\sigma_B - \sigma_A)}{L^2}. \tag{8}$$

The relation derived from (1) is used here: $\partial k/\partial L = -(\sigma_B - \sigma_A)/L^2$.

$$\frac{\partial \varphi_B}{\partial L} = \sigma_A + kL + \frac{\partial k}{\partial L}\cdot\frac{L^2}{2} = = \sigma_A + kL - \frac{(\sigma_B - \sigma_A)}{2} = \frac{(\sigma_A + \sigma_B)}{2}. \tag{9}$$

Thus, on the right side of the clothoid there is a shift and rotation centered at point B, while inside the clothoid we need to take into account only the effect of changing the parameter k.

When changing the curvature of the circle, the parameters of the adjacent clothoids on the left and right change along with the coordinates of the end point of the circle arc and the angle of the tangent in it with the *OX* axis. All this leads to shifts and rotations of the spline part following the end point of the right clothoid. In addition, the coordinates of the internal points of the circle arc, as well as those of the left and right clothoids, also change.

We proceed to derive the formulas that will be used to account for the change in the clothoid parameter.

We will need four integrals:

$$I_1 = \int_0^L \sin\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)tdt,$$

$$I_2 = \int_0^L \cos\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)tdt,$$

$$I_3 = \int_0^L \sin\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)t^2dt,$$

$$I_4 = \int_0^L \cos\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)t^2dt,$$

$$I_1 = \frac{1}{k}\int_0^L \sin\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)(kt + \sigma_A - \sigma_A)dt = = \frac{1}{k}\int_0^L \sin\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)d\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right) - - \frac{\sigma_A}{k}\int_0^L \sin\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)dt = = -\frac{1}{k}(\cos\varphi_B - \cos\varphi_A) - \frac{\sigma_A}{k}(y_B - y_A), \tag{10}$$

$$I_2 = \frac{1}{k}\int_0^L \cos\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)(kt + \sigma_A - \sigma_A)dt = = \frac{1}{k}\int_0^L \cos\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)d\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right) - - \frac{\sigma_A}{k}\int_0^L \cos\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)dt = = \frac{1}{k}(\sin\varphi_B - \sin\varphi_A) - \frac{\sigma_A}{k}(x_B - x_A), \tag{11}$$

$$I_3 = \frac{1}{k}\int_0^L \sin\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)t(kt + \sigma_A - \sigma_A)dt = = -\frac{1}{k}\int_0^L td\cos\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right) - \frac{\sigma_A}{k}I_1 = = -\frac{1}{k}\big(L\cos\varphi_B - (x_B - x_A)\big) + + \frac{\sigma_A}{k^2}\big((\cos\varphi_B - \cos\varphi_A) + \sigma_A(y_B - y_A)\big), \tag{12}$$

$$I_4 = \frac{1}{k}\int_0^L \cos\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right)t(kt + \sigma_A - \sigma_A)dt = = \frac{1}{k}\int_0^L td\sin\left(\varphi_A + \sigma_A t + k\frac{t^2}{2}\right) - \frac{\sigma_A}{k}I_2 = = \frac{1}{k}\big(L\sin\varphi_B - (y_B - y_A)\big) - - \frac{\sigma_A}{k^2}\big((\sin\varphi_B - \sin\varphi_A) - \sigma_A(x_B - x_A)\big). \tag{13}$$

It follows from (5) that

$$\frac{\partial x_B}{\partial k} = -\frac{1}{2}I_3 = \frac{1}{2k}\left(L\cos\varphi_B - (x_B - x_A)\right) - \\ - \frac{\sigma_A}{2k^2}\left((\cos\varphi_B - \cos\varphi_A) + \sigma_A(y_B - y_A)\right), \quad (14)$$

$$\frac{\partial y_B}{\partial k} = \frac{1}{2}I_4 = \frac{1}{2k}\left(L\sin\varphi_B - (y_B - y_A)\right) - \\ - \frac{\sigma_A}{2k^2}\left((\sin\varphi_B - \sin\varphi_A) + \sigma_A(x_B - x_A)\right). \quad (15)$$

For the left clothoid, the curvature at the initial point and the length do not change with the changes in the curvature of the circle. Denoting, as before, the curvature of the circle by σ, taking into account (1) and (5) and fixing σ_A, we obtain:

$$\frac{\partial x_B}{\partial \sigma} = \frac{\partial x_B}{\partial k}\cdot\frac{1}{L}, \quad (16)$$

$$\frac{\partial y_B}{\partial \sigma} = \frac{\partial y_B}{\partial k}\cdot\frac{1}{L}. \quad (17)$$

Let us consider the effect of changing the curvature of the circle σ on the right clothoid. For its initial and end points, as well as its length, we keep the designations A, B, and L, respectively.

$$x_B = \int_0^L \cos\left(\varphi_A + \sigma t + \frac{\sigma_B - \sigma}{L}\cdot\frac{t^2}{2}\right)dt,$$

$$\frac{\partial x_B}{\partial \sigma} = -\int_0^L \sin\left(\varphi_A + \sigma t + \frac{\sigma_B - \sigma}{L}\cdot\frac{t^2}{2}\right)\left(t - \frac{t^2}{2L}\right)dt = \\ = -I_1 + \frac{1}{2L}I_3, \quad (18)$$

$$y_B = \int_0^L \sin\left(\varphi_A + \sigma t + \frac{\sigma_B - \sigma}{L}\cdot\frac{t^2}{2}\right)dt,$$

$$\frac{\partial y_B}{\partial \sigma} = \int_0^L \cos\left(\varphi_A + \sigma t + \frac{\sigma_B - \sigma}{L}\cdot\frac{t^2}{2}\right)\left(t - \frac{t^2}{2L}\right)dt = \\ = I_2 - \frac{1}{2L}I_4. \quad (19)$$

When substituting in (18) and (19) instead of I_1, I_2, I_3, I_4 their values from (10)–(13) for the right clothoid, it should be taken into account that $\sigma_A = \sigma$ and $k = (\sigma_B - \sigma)/L$.

Formulas (18) and (19) can be used in computing the derivatives of the coordinates of any interior point C of the right clothoid along the curvature of the circle by substituting in (10)–(13) x_C and y_C instead of x_B and y_B, φ_C instead of φ_B, and, instead of L, the length of the clothoid from the initial point A to this point C. However, in formulas (18) and (19), L is the length of the entire right clothoid from point A to point B.

Then it follows from (1) and (2) that:

$$\frac{\partial \varphi_B}{\partial \sigma} = \frac{L}{2}. \quad (20)$$

Here σ is the curvature of the circle, φ_B is the angle with the OX axis at the end point of the clothoid, and L is its length.

Formula (20) is applicable to both left and right clothoids.

Now we have everything necessary to proceed to the computation of partial derivatives of displacements on normals (intermediate variables) on basic variables.

4. CALCULATION OF DERIVATIVE DISPLACEMENTS ALONG NORMALS

Thus, it has been found that, even in the presence of clothoids, all spline transformations are reduced to shifts and rotations when changing one main variable. Let us consider how to successively calculate the derivatives of displacements by normals on the main variables without having the corresponding analytical relationships.

4.1. Derivatives by the straight-line length

When changing the length of the line by δl the subsequent part of the spline is shifted in the direction of the changed line. This direction is determined by the angle α of the line with the axis OX (Fig. 3). For the shift along the jth normal, the formula is valid

$$\frac{\partial h_j}{\partial l} = \frac{\sin(\alpha - \beta)}{\sin(\gamma_j - \beta)}, \quad (21)$$

where β is the angle with the OX axis of the tangent (line AB in Fig. 3) to the spline element (in a special case it is a line) at the point of intersection with the jth normal); γ_j is the angle of the normal (C_0C_1 in Fig. 3) with the OX axis.

In Fig. 3, point C is the initial position of the intersection point of the normal and the spline, which corresponds to the value of the intermediate variable h_j. At the shift in the direction determined by the angle α at δl, AB moves to A_1B_1, point C moves to C_2, and C_1 becomes the point of intersection of the normal with the spline. The displacement h_j gets the increment $\delta h_j = CC_1$.

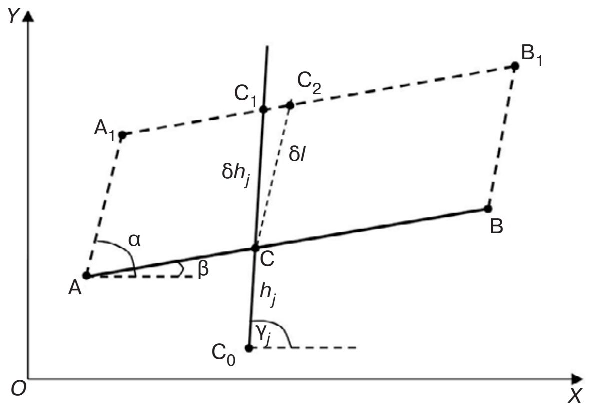

Fig. 3. To the calculation of partial derivatives at the shift

Formula (21) is derived from the sine theorem when applied to the triangle C_1CC_2. It is valid for all normals intersecting the spline to the right of the end of the varying line.

4.2. Derivatives by the arc length of a circle

When the arc length of the circle is changed by δL, the whole subsequent part of the spline (starting from the end point of arc B) is shifted by δL in the direction of the tangent to the circle at B. This direction is determined by the angle α of the tangent with the OX axis. Additionally, the right side of the spline is rotated by the angle $\delta\alpha = \sigma\delta L$ around the point B, where σ is the curvature of the circle. The shift is accounted for in the same way as for the change in the length of a straight line. In [3], the formula (13) for calculating the derivative of the length of the circle of the displacement along the jth normal is given and justified, which, in the notations we have adopted, will be as follows:

$$\frac{\partial h_j}{\partial L} = \frac{\sin(\alpha-\beta)+\left[(x_C-x_B)\cos\beta+(y_C-y_B)\sin\beta\right]\sigma}{\sin(\gamma_j-\beta)}. \quad (22)$$

Here L is the length of the circle arc, α is the angle of the tangent to it at the end point B, x_C, y_C are the coordinates of the point of intersection of the spline with the jth normal, β is the angle of the tangent to the spline at this point C with the OX axis, γ_j is the angle of the normal with the OX axis. This formula takes into account both shift and rotation. It is valid for any normal intersecting the spline to the right of the end point of the circle arc.

4.3. Derivatives by the arc length of the clothoid

When the length of the clothoid changes, the subsequent part of the spline is shifted and rotated with the center at the end point of the clothoid arc B.

The respective increments of displacements along the jth normal to the right of the clothoid are represented in the form:

$$\partial h_j = \partial h_j^s + \partial h_j^r, \quad (23)$$

where ∂h_j^s is the increment of displacement along the jth normal at the shift; ∂h_j^r is the increment of displacement along the jth normal at the rotation.

The change in the coordinates of the end point B occurs due to the tangent shift by ∂L and the change in the parameter k, which additionally leads to a change in the coordinates of the intersection points of the normals with the clothoid.

For the calculation of ∂h_j^s, we use formulas (7) and (8), which give the increments of coordinates ∂x_B and ∂y_B of the end point of the clothoid arc caused by the increment ∂L. The same increments will be given by the shift to the coordinates of all subsequent points. It is notable that the first summands in these formulas correspond to the shift along the tangent at point B by ∂L, while the second summands correspond to the shift due to the change in the parameter of the clothoid while maintaining the curvature at its initial and end points. If we denote the increment h_j^s caused by a shift along the OX axis by ∂x_B via ∂h_{jx}, and along the OY axis by ∂y_B via ∂h_{jy}, then

In order to calculate ∂h_{jx} in formula (21), we replace ∂l by ∂x_B , and α by 0. We obtain $\frac{\partial h_{jx}}{\partial x_B} = -\frac{\sin\beta}{\sin(\gamma_j-\beta)}$.

Similarly for ∂h_{jy} when $\alpha = \pi/2$: $\frac{\partial h_{jy}}{\partial y_B} = \frac{\cos\beta}{\sin(\gamma_j-\beta)}$.

Hence it follows:

$$\partial h_j^s = -\frac{\sin\beta}{\sin(\gamma_j-\beta)}\partial x_B + \frac{\cos\beta}{\sin(\gamma_j-\beta)}\partial y_B. \quad (24)$$

Further:

$$\frac{\partial h_j^s}{\partial L} = -\frac{\sin\beta}{\sin(\gamma_j-\beta)}\cdot\frac{\partial x_B}{\partial L} + \frac{\cos\beta}{\sin(\gamma_j-\beta)}\cdot\frac{\partial y_B}{\partial L}. \quad (25)$$

Derivatives $\frac{\partial x_B}{\partial L}$ and $\frac{\partial y_B}{\partial L}$ are calculated by formulas (7) and (8), in which, instead of $\frac{\partial x_B}{\partial k}$ and $\frac{\partial y_B}{\partial k}$, we should substitute their expressions from (14) and (15), respectively.

For intersections with normals inside the clothoid in (22), instead of (6) and (7), we should use expressions $\frac{\partial x_C}{\partial L} = -\frac{\partial x_C}{\partial k}\cdot\frac{(\sigma_B-\sigma_A)}{L^2}$ and $\frac{\partial y_C}{\partial L} = -\frac{\partial y_C}{\partial k}\cdot\frac{(\sigma_B-\sigma_A)}{L^2}$, while the derivatives $\frac{\partial x_C}{\partial k}$ and $\frac{\partial y_C}{\partial k}$ should be calculated by formulas (13) and (14), substituting x_C, y_C instead of x_B, y_B, and φ_C instead of φ_B and L, representing the length of the clothoid from the initial point A to the end point B.

For the calculation of ∂h_j^r, it is necessary to take into account the rotation of the subsequent part of the spline around the end point of the clothoid B by the angle $\partial\varphi_B$. In [3], the formula for calculating the derivatives of displacements along the jth normal by the rotation angle is derived, which in our notations will be as follows:

$$\frac{\partial h_j^r}{\partial \varphi_B} = \frac{(x_C - x_B)\cos\beta + (y_C - y_B)\sin\beta}{\sin(\gamma_j - \beta)}.$$

Taking into account that, by formula (9) $\partial\varphi_B/\partial L = (\sigma_A + \sigma_B)/2$, we obtain:

$$\frac{\partial h_j^r}{\partial L} = \frac{(x_C - x_B)\cos\beta + (y_C - y_B)\sin\beta}{\sin(\gamma_j - \beta)}(\sigma_A + \sigma_B)/2. \quad (26)$$

Here x_C, y_C are the coordinates of the point of intersection of the spline with the jth normal; β is the angle of the tangent to the spline at this point C with the OX axis; γ_j is the angle of the normal with the OX axis; σ_A and σ_B are the curvature at the initial and end points of the clothoid.

According to (23), the sum of the right parts of (25) and (26) gives the derivative for the subsequent part of the spline.

4.4. Derivatives by curvature

As already mentioned, the most complex transformation of the spline takes place when changing the curvature σ of one circle and maintaining the values of all other main variables: the parameter of the left clothoid is changed, which results in shifts inside it; the right part of the spline is shifted and rotated up to its end; within the circle arc, there are shifts along the normals intersecting it; additionally, there are shifts and rotations of the spline part beyond the end point of the circle arc; finally, the parameter of the right clothoid is changed, which provides shifts and rotations of the spline part beyond the end point of this clothoid, as well as shifts inside it.

We will calculate the derivatives of displacements by normals along the curvature sequentially by sections.

Within the limits of the left clothoid and up to the end of the spline

For the point C of intersection of the jth normal with the clothoid, we designate its coordinates as x_C, y_C, the length from the origin of the arc of the clothoid (point A) to the point C as L_C, the angle of the tangent at the point C as φ_C, and the parameter of the left clothoid as k_1. Let us use formulas (14) and (15):

$$\frac{\partial x_C}{\partial k_1} = \frac{1}{2k_1}\left(L_C\cos\varphi_C - (x_C - x_A)\right) - \frac{\sigma_A}{2k_1^2}\left((\cos\varphi_C - \cos\varphi_A) + \sigma_A(y_C - y_A)\right), \quad (27)$$

$$\frac{\partial y_C}{\partial k_1} = \frac{1}{2k_1}\left(L_C\sin\varphi_C - (y_C - y_A)\right) - \frac{\sigma_A}{2k_1^2}\left((\sin\varphi_B - \sin\varphi_A) + \sigma_A(x_C - x_A)\right). \quad (28)$$

According to (16) and (17), $\frac{\partial x_C}{\partial \sigma} = \frac{\partial x_C}{\partial k_1}\cdot\frac{1}{L}$ and $\frac{\partial y_C}{\partial \sigma} = \frac{\partial y_C}{\partial k_1}\cdot\frac{1}{L}$, where L is the length of the clothoid AB.

According to (24), the coordinate increment gives the normal displacement increment by $\partial h_j^s = -\frac{\sin\beta}{\sin(\gamma_j - \beta)}\partial x_C + \frac{\cos\beta}{\sin(\gamma_j - \beta)}\partial y_C$. For the corresponding derivative on the curvature of the circle of the jth normal displacement within the left clothoid, we obtain:

$$\frac{\partial h_j^s}{\partial \sigma} = \left(-\frac{\sin\beta}{\sin(\gamma_j - \beta)}\cdot\frac{\partial x_C}{\partial k_1} + \frac{\cos\beta}{\sin(\gamma_j - \beta)}\cdot\frac{\partial y_C}{\partial k_1}\right)\Big/ L. \quad (29)$$

Here, as before, β is the angle with the OX axis of the tangent to the spline at the point C of intersection with the normal, while γ_j is the angle of the normal with the OX axis. In (29), it is necessary to substitute both $\frac{\partial x_C}{\partial k_1}$ and from (27) and (28).

For the normals intersecting the spline to the right of the left clothoid, formulas (27) and (28) are applied to the end point B and the is result substituted into formula (29), in which the angles β and γ_j refer to the corresponding normal. For the same normals, the derivative $\frac{\partial h_j^r}{\partial \sigma}$ due to the rotation of the tangent at point B of the clothoid is calculated using (26) and $\frac{\partial \varphi_B}{\partial \sigma} = \frac{L}{2}$, which is derived from (4). As a result, for an arbitrary point C of the intersection of the normal with the spline to the right of the left clothoid, we obtain:

$$\frac{\partial h_j^r}{\partial \sigma} = \frac{(x_C - x_B)\cos\beta + (y_C - y_B)\sin\beta}{\sin(\gamma_j - \beta)}\cdot\frac{L}{2}. \quad (30)$$

Summarizing (29) and (30) gives

$$\frac{\partial h_j}{\partial \sigma} = \frac{\partial h_j^{\mathrm{s}}}{\partial \sigma} + \frac{\partial h_j^{\mathrm{r}}}{\partial \sigma}. \qquad (31)$$

Within the circle and to the end of the spline

Additionally, there are changes in coordinates of intersection points with normals within the circle arc due to changes in its curvature.

Formula (17), obtained in [1] for calculating the partial derivatives of the displacements h_j along normals within the circle by radius R, has the following form $\frac{\delta h_j}{\delta R} = \frac{\cos(\beta - \alpha) - 1}{\sin(\gamma - \beta)}$. In it, there are the angles with the OX axis: α and β are the angles of the tangents to the arc of the circle at its initial and end points, while γ_j is the angle of the jth normal. In our designations for the curvature derivative of the displacements along the normals inside the circle, we obtain:

$$\frac{\partial h_j^{\mathrm{s1}}}{\partial \sigma} = \frac{1 - \cos(\beta - \varphi_{\mathrm{B}})}{\sin(\gamma_j - \beta)\sigma^2}. \qquad (32)$$

As a result, for the derivatives of displacements along the normals intersecting the arc of the circle, we obtain:

$$\frac{\partial h_j}{\partial \sigma} = \frac{\partial h_j^{\mathrm{s}}}{\partial \sigma} + \frac{\partial h_j^{\mathrm{r}}}{\partial \sigma} + \frac{\partial h_j^{\mathrm{s1}}}{\partial \sigma}. \qquad (33)$$

Due to the change of curvature σ, there is an additional shift of the whole subsequent part of the spline from the end of the circle arc (point B), as well as its rotation centered at this point.

According to formulas (14) and (15) from [1], when passing from radius to curvature, we obtain:

$$\frac{\partial x_{\mathrm{B}}}{\partial \sigma} = -\frac{\sin\beta - \sin\alpha - (\beta - \alpha)\cos\beta}{\sigma^2}, \qquad (34)$$

$$\frac{\partial y_{\mathrm{B}}}{\partial \sigma} = -\frac{\cos\alpha - \cos\beta - (\beta - \alpha)\sin\beta}{\sigma^2}. \qquad (35)$$

Here, α and β are the angles with the OX axis of the tangents to the arc of the circle at its initial and end points, respectively.

We obtain for the derivatives of displacements h_j^{s2} along the normals resulting from the shift at the end point of the circle using formula (24), which allows us to switch from displacements along x and y coordinates to displacements along the normal:

$$\frac{\partial h_j^{\mathrm{s2}}}{\partial \sigma} = -\frac{\sin\beta_1}{\sin(\gamma_j - \beta_1)} \cdot \frac{\partial x_{\mathrm{B}}}{\partial \sigma} + \frac{\cos\beta_1}{\sin(\gamma_j - \beta_1)} \cdot \frac{\partial y_{\mathrm{B}}}{\partial \sigma}. \qquad (36)$$

Here, β_1 is the angle with the OX axis of the tangent to the spline at the point of its intersection by the jth normal; γ_j is the angle of this normal with the OX axis.

Following substitution $\frac{\partial x_{\mathrm{B}}}{\partial \sigma}$ and $\frac{\partial y_{\mathrm{B}}}{\partial \sigma}$ from (34) and (35) into (36) and simplifications, we obtain:

$$\frac{\partial h_j^{\mathrm{s2}}}{\partial \sigma} = -\frac{\cos(\beta_1 - \alpha) - \cos(\beta_1 - \beta) + (\beta - \alpha)\sin(\beta_1 - \beta)}{\sin(\gamma_j - \beta_1)\sigma^2}. \qquad (37)$$

The consequences of the tangent rotation at the end point of the circle when its curvature changes will be taken into account in the same way as was done above for the tangent rotation at the end of the left clothoid. According to (26)

$$\frac{\partial h_j^{\mathrm{r2}}}{\partial \varphi_{\mathrm{B}}} = \frac{(x_{\mathrm{C}} - x_{\mathrm{B}})\cos\beta + (y_{\mathrm{C}} - y_{\mathrm{B}})\sin\beta}{\sin(\gamma_j - \beta)}.$$

Here x_{C}, y_{C} are the coordinates of the point of intersection of the spline with the jth normal; β is the angle of the tangent to the spline at this point C with the OX axis; γ_j is the angle of the normal with the OX axis; φ_{B} is the angle of the tangent to the arc of the circle at its end point.

Taking into account that, for a circle $\frac{\partial \varphi_{\mathrm{B}}}{\partial \sigma} = L$, where L is the length of the circle arc, we obtain:

$$\frac{\partial h_j^{\mathrm{r2}}}{\partial \sigma} = \frac{(x_{\mathrm{C}} - x_{\mathrm{B}})\cos\beta + (y_{\mathrm{C}} - y_{\mathrm{B}})\sin\beta}{\sin(\gamma_j - \beta)} L. \qquad (38)$$

Formulas (37) and (38) are true for all points of intersection of normals with the spline not only within the right clothoid, but also up to the end of the spline. The effect of a change in the circle curvature on the right clothoid is taken into account in the same way.

5. CALCULATION OF THE OBJECTIVE FUNCTION GRADIENT

The initial approximation for the optimization algorithm is a spline obtained by a separate program implementing the dynamic programming method [1]. Using this spline, the offsets of given survey points along the normals to the design position are determined (Fig. 2). These are the current values of intermediate variables h_j. In order to determine them, the elements of the spline starting from the initial straight line are sequentially considered. For each element (line, clothoid, circle) the number of the first normal intersecting it is memorized.

Formula (9) from [1] is used for determining the intersection points of normals with the circle. The iterative algorithm [19] is used to find intersections with the clothoid. Then, for each basic variable x_i (lengths of elements and curvatures of circles), the number of the first normal j_i is sequentially determined, the displacement along which is affected by the change of the corresponding basic variable. For the lengths of straight lines and circles, this is the number of the first normal that intersects the next element. For the length of a clothoid, it is the number of the first normal intersecting it; for the curvature of a circle, it is the number of the first normal intersecting the left clothoid. The number of the final normal for all elements is the number of the last normal n.

The derivatives of the initial objective function (2) by main variables are calculated by the formula:

$$\frac{\partial F(\mathbf{h}(\mathbf{x}))}{\partial x_i} = \sum_{j=j_i}^{n} h_j \frac{\partial h_j}{\partial x_j}. \tag{39}$$

Here $\mathbf{x}$ and $\mathbf{h}$ are the vectors of basic and intermediate variables, respectively.

The same modified Lagrange function [20–22] and the same algorithm [23, 24] as for the spline consisting of line segments and arcs of circles [3] are used to optimize the spline parameters. For this purpose, the derivative of the penalty function [3] is added to the right part of (39) when calculating the gradient.

CONCLUSIONS

The main result of this research is the development of mathematical models and algorithms for approximation of functions given by a discrete sequence of points by compound splines of complex structure, including splines with clothoids. A successful choice of variables allowed us to solve the task of approximating multivalued functions. Such problems are typical of those arising in the design of railroad and highway traces.

The unique approach of obtaining formulas for calculating partial derivatives in the absence of analytical expressions of differentiable functions can also be used in solving other problems.

Performed calculations using the experimental programs have shown that, although the presence of clothoids significantly increases counting time, this does not become critical when using commonly available modern personal computers. Unfortunately, the developed algorithms and programs have yet to find practical application due to the lack of interest in improving the quality of design solutions at the same time as reducing costs in the construction and reconstruction of the roads and railways.

Authors' contribution. All authors equally contributed to the present work.

REFERENCES

1. Karpov D.A., Struchenkov V.I. Two-stage spline-approximation in linear structure routing. *Russ. Technol. J.* 2021;9(5):45−56 (in Russ.). https://doi.org/10.32362/2500-316X-2021-9-5-45-56
2. Karpov D.A., Struchenkov V.I. Spline approximation of multivalued function in leaner structures routing. *Russ. Technol. J.* 2022;10(4):65–74 (in Russ.). https://doi.org/10.32362/2500-316X-2022-10-4-65-74
3. Karpov D.A., Struchenkov V.I. Optimization of spline parameters in approximation of multi-valued function *Russ. Technol. J.* 2023;11(2):72–83 (in Russ.). https://doi.org/10.32362/2500-316X-2023-11-2-72-83
4. Li W., Pu H., Schonfeld P., et al. A Method for Automatically Recreating the Horizontal Alignment Geometry of Existing Railways. *Comput. Aided Civ. Infrastruct. Eng.* 2019;34(1):71–94. https://doi.org/10.1111/mice.12392
5. Jha M.K., McCall C., Schonfeld P. Using GIS, genetic algorithms, and visualization in highway development. *Comput. Aided Civ. Infrastruct. Eng.* 2001;16(6):399–414. https://doi.org/10.1111/0885-9507.00242
6. Jha M.K., Schonfeld P. A highway alignment optimization model using geographic information systems. *Transp. Res. Part A. Policy Pract*. 2004;8(6):455–481. https://doi.org/10.1016/j.tra.2004.04.001
7. Jong J.C., Jha M.K., Schonfeld P. Preliminary highway design with genetic algorithms and geographic information systems. *Comput. Aided Civ. Infrastruct. Eng.* 2000;15(4):261–271. https://doi.org/10.1111/0885-9507.00190
8. Kang M.W., Schonfeld P., Yang N. Prescreening and repairing in a genetic algorithm for highway alignment optimization. *Comput. Aided Civ. Infrastruct. Eng.* 2009;24(2):109–119. https://doi.org/10.1111/j.1467-8667.2008.00574.x
9. Pushak Y., Hare W., Lucet Y. Multiple-path selection for new highway alignments using discrete algorithms. *Eur. J. Oper. Res*. 2016;248(2):415–427. https://doi.org/10.1016/j.ejor.2015.07.039
10. Sarma K.C., Adeli H. Bilevel parallel genetic algorithms for optimization of large steel structures. *Comput. Aided Civ. Infrastruct. Eng.* 2001;16(5):295–304. https://doi.org/10.1111/0885-9507.00234
11. Shafahi Y., Bagherian M. A customized particle swarm method to solve highway alignment optimization problem. *Comput. Aided Civ. Infrastruct. Eng.* 2013;28(1):52–67. https://doi.org/10.1111/j.1467-8667.2012.00769.x
12. Bosurgi G., D'Andrea A. A polynomial parametric curve (PPC-curve) for the design of horizontal geometry of highways. *Comput. Aided Civ. Infrastruct. Eng.* 2012;27(4):303–312. https://doi.org/10.1111/j.1467-8667.2011.00750.x
13. Cerf R. The quasispecies regime for the simple genetic algorithm with roulette wheel Selection. Cornell University. *Adv. Appl. Probability*. 2017;49(3):903–926. https://doi.org/10.1017/apr.2017.26

14. Oudshoorn M., Koppenberg T., Yorke-Smith N. Optimization of annual planned rail maintenance. *Comput. Aided Civ. Infrastruct. Eng.* 2021;37(6):669–687. https://doi.org/10.1111/mice.12764
15. Struchenkov V.I. New algorithm for perelement calculation of line structures routes. *Informacionnye tekhnologii = Information Technologies*. 2015;21(4):271–276 (in Russ.).
16. Borodakii Yu.V., Zagrebaev A.M., Kritsyna N.A., Kulyabichev Yu.P., Shumilov Yu.Yu. *Nelineinoe programmirovanie v sovremennykh zadachakh optimizatsii* (*Nonlinear Programming in Modern Optimization Problem*). Moscow: NIYAU MEPhI; 2008. 244 p. (in Russ.).
17. Polyakov V.M., Agalarov Z.S. *Metody optimizatsii* (*Optimization Methods)*. Moscow: Dashkov i K; 2022. 86 p. (in Russ.). ISBN 978-5-3940-5003-9
18. Gasnikov A.V. *Sovremennye chislennye metody optimizatsii. Metod universal'nogo gradientnogo spuska* (*Modern Numerical Optimization Methods. Universal Gradient Method Descent*). Moscow: MTsNMO; 2021.272 p. (in Russ.). ISBN 978-5-4439-1614-9
19. Struchenkov V.I. *Metody optimizatsii trass v SAPR lineinykh sooruzhenii* (*Methods for Route Optimization in CAD of Linear Structures*). Moscow: SOLON-Press; 2020. 272 p. (in Russ.). ISBN 978-5-9135-9139-5
20. Kochenderfer M.D., Wheeler T.A. *Algoritmy optimizatsii* (*Algorithms for Optimization*). Moscow: Vil'yams; 2020. 528 p. (in Russ.).
[Kochenderfer M.D., Wheeler T.A. *Algorithms for Optimization*. London: The MIT Press; 2019. 520 p.]
21. Audet C., Hare W. *Derivative-Free and Blackbox Optimization.* Springer Series in Operations Research and Financial Engineering. Springer; 2017. 302 p. https://doi.org/10.1007/978-3-319-68913-5
22. Chernorutskii I.G. Metody optimizatsii. Komp'yuternye tekhnologii (*Methods of Optimization*. *Computer Technologies*). St. Petersburg: BHV-Petersburg; 2011. 329 p. (in Russ.).
23. Gill Ph.E., Murray W., Wright M.H. *Prakticheskaya optimizatsiya* (*Practical Optimization*): transl. from Engl. Moscow: Mir; 1985. 509 p. (in Russ.).
[Gill Ph.E., Murray W., Wright M.H. *Practical Optimization*. London: Academic Press; 1981. 402 p.]
24. Larichev O.I., Gorvits G.G. *Metody poiska lokal'nykh ekstremumov ovrazhnykh funktsii* (*Methods for Finding Local Extrema of Ravine Functions*). Moscow: Nauka; 1990. 96 p. (in Russ.).

СПИСОК ЛИТЕРАТУРЫ

1. Карпов Д.А., Струченков В.И. Двухэтапная сплайн-аппроксимация в компьютерном проектировании трасс линейных сооружений. *Russ. Technol. J.* 2021;9(5):45–56. https://doi.org/10.32362/2500-316X-2021-9-5-45-56
2. Карпов Д.А., Струченков В.И. Сплайн-аппроксимация многозначных функций в проектировании трасс линейных сооружений. *Russ. Technol. J.* 2022;10(4):65–74. https://doi.org/10.32362/2500-316X-2022-10-4-65-74
3. Карпов Д.А., Струченков В.И. Оптимизация параметров сплайна при аппроксимации многозначных функций. *Russ. Technol. J.* 2023;11(2);72–83. https://doi.org/10.32362/2500-316X-2023-11-2-72-83
4. Li W., Pu H., Schonfeld P., et al. A Method for Automatically Recreating the Horizontal Alignment Geometry of Existing Railways. *Comput. Aided Civ. Infrastruct. Eng.* 2019;34(1):71–94. https://doi.org/10.1111/mice.12392
5. Jha M.K., McCall C., Schonfeld P. Using GIS, genetic algorithms, and visualization in highway development. *Comput. Aided Civ. Infrastruct. Eng.* 2001;16(6):399–414. https://doi.org/10.1111/0885-9507.00242
6. Jha M.K., Schonfeld P. A highway alignment optimization model using geographic information systems. *Transp. Res. Part A. Policy Pract*. 2004;8(6):455–481. https://doi.org/10.1016/j.tra.2004.04.001
7. Jong J.C., Jha M.K., Schonfeld P. Preliminary highway design with genetic algorithms and geographic information systems. *Comput. Aided Civ. Infrastruct. Eng.* 2000;15(4):261–271. https://doi.org/10.1111/0885-9507.00190
8. Kang M.W., Schonfeld P., Yang N. Prescreening and repairing in a genetic algorithm for highway alignment optimization. *Comput. Aided Civ. Infrastruct. Eng.* 2009;24(2):109–119. https://doi.org/10.1111/j.1467-8667.2008.00574.x
9. Pushak Y., Hare W., Lucet Y. Multiple-path selection for new highway alignments using discrete algorithms. *Eur. J. Oper. Res*. 2016;248(2):415–427. https://doi.org/10.1016/j.ejor.2015.07.039
10. Sarma K.C., Adeli H. Bilevel parallel genetic algorithms for optimization of large steel structures. *Comput. Aided Civ. Infrastruct. Eng.* 2001;16(5):295–304. https://doi.org/10.1111/0885-9507.00234
11. Shafahi Y., Bagherian M. A customized particle swarm method to solve highway alignment optimization problem. *Comput. Aided Civ. Infrastruct. Eng.* 2013;28(1):52–67. https://doi.org/10.1111/j.1467-8667.2012.00769.x
12. Bosurgi G., D'Andrea A. A polynomial parametric curve (PPC-curve) for the design of horizontal geometry of highways. *Comput. Aided Civ. Infrastruct. Eng.* 2012;27(4):303–312. https://doi.org/10.1111/j.1467-8667.2011.00750.x
13. Cerf R. The quasispecies regime for the simple genetic algorithm with roulette wheel Selection. Cornell University. *Adv. Appl. Probability*. 2017;49(3):903–926. https://doi.org/10.1017/apr.2017.26
14. Oudshoorn M., Koppenberg T., Yorke-Smith N. Optimization of annual planned rail maintenance. *Comput. Aided Civ. Infrastruct. Eng.* 2021;37(6):669–687. https://doi.org/10.1111/mice.12764
15. Струченков В.И. Новый алгоритм поэлементного расчета трасс в САПР линейных сооружений. *Информационные технологии.* 2015;21(4):271–276.
16. Бородакий Ю.В., Загребаев А.М., Крицына Н.А., Кулябичев Ю.П., Шумилов Ю.Ю. *Нелинейное программирование в современных задачах оптимизации*. М.: НИЯУ МИФИ; 2011. 244 с. ISBN 987-5-7262-1451-1

17. Поляков В.М., Агаларов З.С. *Методы оптимизации*. М.: Дашков и К; 2022. 86 с. ISBN 978-5-3940-5003-9
18. Гасников А.В. *Современные численные методы оптимизации. Метод универсального градиентного спуска*. М.: МЦНМО; 2021. 272 с. ISBN 978-5-4439-1614-9
19. Струченков В.И. *Методы оптимизации трасс в САПР линейных сооружений*. М.: СОЛОН-Пресс; 2020. 272 с. ISBN 978-5-9135-9139-5
20. Кохендерфер М. Д., Уилер Т.А. *Алгоритмы оптимизации*. М.: Вильямс; 2020. 528 с.
21. Audet C., Hare W. *Derivative-Free and Blackbox Optimization.* Springer Series in Operations Research and Financial Engineering. Springer; 2017. 302 p. https://doi.org/10.1007/978-3-319-68913-5
22. Черноруцкий И.Г. *Методы оптимизации. Компьютерные технологии*. СПб.: БХВ-Петербург; 2011. 329 с.
23. Гилл Ф., Мюррей У., Райт М. *Практическая оптимизация*: пер. с англ. М.: Мир; 1985. 509 с.
24. Ларичев О.И., Горвиц Г.Г. *Методы поиска локальных экстремумов овражных функций*. М.: Наука; 1990. 96 с.

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