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RESEARCH ARTICLE

Properties of the Wrońskian determinant of a system of solutions to a linear homogeneous equation: The case when the number of solutions is less than the order of the equation

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Abstract
Objectives. The work sets out to study the properties of the Wrońskian determinant of the system of solutions to a linear homogeneous equation in cases when the number of solutions is less than the order of the equation, comparing them with the known properties of the same determinant when the number of solutions is equal to the order of the equation.
Methods. The work uses the methods of linear algebra according to the theory of ordinary differential equations, as well as mathematical and complex analysis.
Results. It is shown that the vanishing of a considered determinant on an arbitrarily small interval implies its vanishing on the entire domain of definition; the solutions turn out to be linearly dependent. A stronger result is obtained in three cases: (1) if the coefficients of the equation are analytic functions; (2) if the number of solutions is equal to one; (3) if the number of solutions is one less than the order of the equation. Namely, if the set of zeros of the considered Wrońskian has a limit point belonging to the domain of definition of solutions, then the determinant is identically equal to zero and the solutions are linearly dependent.
Conclusions. According to the obtained results, the Wrońskian of a system of solutions of a linear homogeneous equation can serve as an indicator of the linear dependence or independence of this system in cases where the number of solutions is lower than the order of the equation; here, the solutions are linearly dependent if and only if their Wrońskian is identically equal to zero. In this case, there is no need to check whether the determinant vanishes over the entire domain of definition, since it is sufficient to do this on an arbitrarily chosen interval or even (in the special cases listed above) on an arbitrarily chosen set having a limit point.

Keywords: linear homogeneous differential equation, Wrońskian, zeros of the Wrońskian, linear dependence, linear independence

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НАУЧНАЯ СТАТЬЯ

Свойства определителя Вронского системы решений линейного однородного уравнения: случай, когда число решений меньше порядка уравнения

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Резюме
Цели. Целью работы является изучение свойств определителя Вронского системы решений линейного однородного дифференциального уравнения в случае, когда число решений меньше порядка уравнения, и сравнение их с известными свойствами такого же определителя, но в случае равенства числа решений порядку уравнения.
Методы. В работе использованы методы линейной алгебры и теории обыкновенных дифференциальных уравнений, а также математического и комплексного анализа.
Результаты. Показано, что обращение в нуль рассматриваемого определителя на сколь угодно малом интервале влечет за собой обращение его в нуль на всей области определения, а решения при этом оказываются линейно зависимыми. В трех случаях: 1) если коэффициенты уравнения являются аналитическими функциями, 2) если число решений равно единице и 3) если число решений на единицу меньше порядка уравнения – получен более сильный результат. Именно, если множество нулей рассматриваемого определителя Вронского имеет предельную точку, принадлежащую области определения решений, то определитель тождественно равен нулю и решения линейно зависимы.
Выводы. Полученные результаты означают, что определитель Вронского системы решений линейного однородного уравнения в ситуации, когда число решений меньше порядка уравнения, служит индикатором линейной зависимости или независимости этой системы: решения линейно зависимы тогда и только тогда, когда их определитель Вронского тождественно равен нулю. При этом нет необходимости проверять обращение определителя в нуль на всей области определения, достаточно сделать это на произвольно выбранном интервале или даже (в перечисленных выше частных случаях) на произвольно выбранном множестве, имеющем предельную точку.

Ключевые слова: линейное однородное дифференциальное уравнение, определитель Вронского, нули определителя Вронского, линейная зависимость, линейная независимость

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INTRODUCTION

One of the main tools of mathematical modeling is ordinary differential equations, which serve as models for describing a wide variety of phenomena and processes [1–5]. In turn, an important tool for studying differential equations, primarily in terms of checking the linear dependence or independence of their solutions, is the Wrońskian.

It should be recalled that the Wrońskian of the system of functions $y_1(x), y_2(x), \ldots, y_k(x)$, $x \in (a, b)$ consists in the following function:

$$W(x) = W_{y_1,y_2,\ldots,y_k}(x) = \\ = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_k(x) \\ y_1'(x) & y_2'(x) & \ldots & y_k'(x) \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(k-1)}(x) & y_2^{(k-1)}(x) & \ldots & y_k^{(k-1)}(x) \end{vmatrix}. \quad (1)$$

The Wrońskian determinant theory is presented in practically every textbook on ordinary differential equations [6–12]; with few exceptions, such determinants are based on n solutions to the nth-order linear homogeneous equation. The remarkable properties of such determinants permit their use as indicators of linear dependence or independence of the considered system of solutions. State more precisely, a system of n solutions to the nth-order linear homogeneous equation is linearly dependent if and only if the Wrońskian of this system is identically equal to zero, and is linearly independent if and only if its Wrońskian is not equal to zero at any point in the definition domain of the considered solutions. It is also well-understood (see the corresponding examples in [6, 9, and 12]) that the situation is quite different for a Wrońskian of a system of functions not presenting solutions to a linear homogeneous equation; the determinant can be zero, even identically, if the functions are linearly independent. Otherwise stated, if the Wrońskian of some system of functions turns out to be identically equal to zero, no answer to the question about the linear dependence or independence of this system could be obtained by this means.

Here, a natural question arises: would the "good" properties of the Wrońskian of a system of solutions to the nth-order linear homogeneous equation be preserved if we take $k < n$ instead of n solutions? If so, could the Wrońskian of such a system be used just as effectively to find out its linear dependence or independence? This question is formulated, for example, in [9]; here however, the study of this question is limited to an example showing that the Wrońskian of a linearly independent system of $k < n$ solutions (in the example, $k = 2$ and $n = 3$), unlike that of a linearly independent system of n solutions, can go to zero at some points of its definition domain.

It is not difficult to provide an example for arbitrary n and $k < n$, when the Wrońskian of a linearly independent system of solutions goes to zero even in an infinite set of points. In what follows, we shall consider equation $y^{(n)} + y^{(n-2)} = 0$, whose fundamental system of solutions consists of the functions $1, x, x^2, \ldots, x^{n-3}, \sin x, \cos x$. For arbitrary $k \le n - 1$, we take the set of k solutions $1, x, x^2, \ldots, x^{k-2}, \sin x$ (for $k = 1$, we take one solution $\sin x$), whose Wrońskian coincides either with $\sin x$ or with $\cos x$ to the nearest numerical factor and consequently has an infinite number of zeros on the numerical axis.

The Wrońskian of a system from the $(n - 1)$th solution to the nth-order linear homogeneous equation is studied in [13], where, in particular, it is shown that, in the case of linear independence of such system, its Wrońskian cannot have an infinite number of zeros on any finite segment. In the cited work, the case of arbitrary number of solutions less than the order of the equation n is considered. The main result is contained in Theorem 1 stating that the equality to zero of the Wrońskian of such system of solutions on any interval implies its linear dependence. Thus, in the case of linear independence of solutions, the Wrońskian cannot be zero on any interval, even an arbitrarily small one.

Theorem 2 shows that this result can be strengthened in a number of special cases, including the above-mentioned case $k = n - 1$. Precisely stated, the set of zeros of the Wrońskian of a linearly independent system of solutions cannot have limit points in its definition domain, and, hence, cannot have an infinite number of zeros on any finite segment. For the case $k = n - 1$, a proof different from that given in [13], which allows weakening the conditions on the coefficients of the equation, is provided.

Thus, it can be said that the Wrońskian of a system of $k < n$ solutions to the nth-order linear homogeneous equation by its properties occupies an intermediate position between the Wrońskian determinant of an arbitrary system of functions and that of a system of n solutions, according to which properties such a determinant can be used to find out whether the considered system of solutions is linearly dependent or independent.

MAIN RESULT

We shall consider the following nth-order linear homogeneous equation:

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0, \quad (2)$$

whose coefficients are $a_{n-1}(x), \ldots, a_0(x) \in C(a, b)$, $-\infty \le a < b \le +\infty$.

As is known, any solution to such equation continues over the entire interval (a, b). In the following, only solutions defined on (a, b) are considered.

Let $y_1(x), y_2(x), \ldots, y_k(x)$ be solutions to Eq. (2), $k \le n - 1$, $W_{y_1,y_2,\ldots,y_k}(x)$ be their Wrońskian (1). We shall also consider determinants of the following form:

$$W^{\alpha_1,\alpha_2,\ldots,\alpha_k}_{y_1,y_2,\ldots,y_k}(x) = \begin{vmatrix} y_1^{(\alpha_1)}(x) & y_2^{(\alpha_1)}(x) & \ldots & y_k^{(\alpha_1)}(x) \\ y_1^{(\alpha_2)}(x) & y_2^{(\alpha_2)}(x) & \ldots & y_k^{(\alpha_2)}(x) \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(\alpha_k)}(x) & y_2^{(\alpha_k)}(x) & \ldots & y_k^{(\alpha_k)}(x) \end{vmatrix},$$

where $0 \le \alpha_1, \alpha_2, \ldots, \alpha_k \le n$.

It is clear that if among the numbers $\alpha_1, \ldots, \alpha_k$ there are coincident ones, then $W^{\alpha_1,\alpha_2,\ldots,\alpha_k}_{y_1,y_2,\ldots,y_k}(x) \equiv 0$. The determinants $W^{\alpha_1,\alpha_2,\ldots,\alpha_k}_{y_1,y_2,\ldots,y_k}$ with $0 \le \alpha_1 < \alpha_2 < \ldots < \alpha_k \le n-1$ will be called *generalized Wrońskians*. Obviously, there are exactly C_n^k of different generalized Wrońskians of the system of k solutions including the Wrońskian $W_{y_1,y_2,\ldots,y_k} = W^{0,1,\ldots,k-1}_{y_1,y_2,\ldots,y_k}$ itself. It should be noted that in the case of linear dependence of the system of solutions $y_1, y_2, \ldots, y_k$ on some interval $(\alpha, \beta) \subset (a, b)$ (and hence, due to the uniqueness theorem of the solution to the Cauchy problem, on the entire interval (a, b)), all its generalized Wrońskians are identically zero on (a, b).

LEMMA 1. Let $y_1, y_2, \ldots, y_k$, $k \le n-1$ be solutions to Eq. (2), and let determinant $W_{y_1,y_2,\ldots,y_k}(x)$ be identically zero on some interval $(\alpha, \beta) \subset (a, b)$. Then for an arbitrary solution y_{k+1} all generalized Wrońskians of the system of solutions $y_1, y_2, \ldots, y_{k+1}$ are identically zero on (a, b).

The PROOF is carried out by induction on k. For $k = 1$, the statement is true due to the uniqueness theorem of the solution to the Cauchy problem. Let us assume it is true for some $k \le n-2$, and prove its validity for $k+1$.

Let $y_1, y_2, \ldots, y_{k+1}$ be solutions to Eq. (2) and

$$W_{y_1,\ldots,y_{k+1}}(x) = 0 \quad \forall x \in (\alpha,\beta) \subset (a,b). \tag{3}$$

We shall take the one less order determinant $W_{y_1,\ldots,y_k}(x)$. There are two possible cases: either $W_{y_1,\ldots,y_k}(x) \equiv 0$ on the interval (α, β) or $x_0 \in (\alpha, \beta)$ $W_{y_1,\ldots,y_k}(x_0) \ne 0$ at some point. We shall consider these cases.

1. Let $W_{y_1,\ldots,y_k}(x) = 0 \ \forall x \in (\alpha,\beta)$. Then by inductive assumption, all generalized Wrońskians of solutions $y_1, y_2, \ldots, y_{k+1}$ are identically zero on (a, b). Take an arbitrary solution y_{k+2} and consider the generalized Wrońskian $W^{\alpha_1,\ldots,\alpha_{k+2}}_{y_1,\ldots,y_{k+2}}$. Decomposing it by the last column, we obtain:

$$W^{\alpha_1,\ldots,\alpha_{k+2}}_{y_1,\ldots,y_{k+2}}(x) =$$

$$= \begin{vmatrix} y_1^{(\alpha_1)}(x) & y_2^{(\alpha_1)}(x) & \ldots & y_{k+2}^{(\alpha_1)}(x) \\ y_1^{(\alpha_2)}(x) & y_2^{(\alpha_2)}(x) & \ldots & y_{k+2}^{(\alpha_2)}(x) \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(\alpha_{k+2})}(x) & y_2^{(\alpha_{k+2})}(x) & \ldots & y_{k+2}^{(\alpha_{k+2})}(x) \end{vmatrix} =$$

$$= (-1)^{k+3} y_{k+2}^{(\alpha_1)}(x) W^{\alpha_2,\ldots,\alpha_{k+2}}_{y_1,\ldots,y_{k+1}}(x) +$$

$$+ (-1)^{k+4} y_{k+2}^{(\alpha_2)}(x) W^{\alpha_1,\alpha_3,\ldots,\alpha_{k+2}}_{y_1,\ldots,y_{k+1}}(x) + \ldots +$$

$$+ y_{k+2}^{(\alpha_{k+2})}(x) W^{\alpha_1,\ldots,\alpha_{k+1}}_{y_1,\ldots,y_{k+1}}(x) = 0 \quad \forall x \in (a,b),$$

and thus, the statement of the lemma is proved.

2. Now let $W_{y_1,\ldots,y_k}(x_0) \ne 0$ at point $x_0 \in (\alpha, \beta)$. Then due to continuity, $W_{y_1,\ldots,y_k}(x) \ne 0$ on some interval (α_1, β_1) containing point x_0. We shall show that the functions $y_1, y_2, \ldots, y_{k+1}$ are linearly dependent on this interval.

Due to (3), the columns of determinant $W_{y_1,\ldots,y_{k+1}}(x)$ are linearly dependent at each point $x \in (\alpha, \beta)$, i.e., there exist constants $\lambda_1(x), \lambda_2(x), \ldots, \lambda_{k+1}(x)$ being not equal to zero simultaneously (in general, different for each point x), such that

$$\begin{cases} \lambda_1(x) y_1(x) + \ldots + \lambda_{k+1}(x) y_{k+1}(x) = 0, \\ \ldots \\ \lambda_1(x) y_1^{(k)}(x) + \ldots + \lambda_{k+1}(x) y_{k+1}^{(k)}(x) = 0. \end{cases} \tag{4}$$

It should be noted that at $x \in (\alpha_1, \beta_1)$ $\lambda_{k+1}(x) \ne 0$. Indeed, otherwise, the following is obtained from the first k equalities (4):

$$\begin{cases} \lambda_1(x) y_1(x) + \ldots + \lambda_k(x) y_k(x) = 0, \\ \ldots \\ \lambda_1(x) y_1^{(k-1)}(x) + \ldots + \lambda_k(x) y_k^{(k-1)}(x) = 0, \end{cases} \tag{5}$$

from which $\lambda_1(x) = \lambda_2(x) = \ldots = \lambda_k(x) = 0$, since the determinant of system (5) is determinant $W_{y_1,\ldots,y_k}(x)$ not equal to zero on the interval (α_1, β_1).

For each $x \in (\alpha_1, \beta_1)$, we shall solve the equations of system (4) with respect to the function y_{k+1} and its derivatives, as follows:

$$\begin{cases} y_{k+1}(x) = \mu_1(x) y_1(x) + \ldots + \mu_k(x) y_k(x), \\ y'_{k+1}(x) = \mu_1(x) y'_1(x) + \ldots + \mu_k(x) y'_k(x), \\ \ldots \\ y_{k+1}^{(k)}(x) = \mu_1(x) y_1^{(k)}(x) + \ldots + \mu_k(x) y_k^{(k)}(x), \end{cases} \tag{6}$$

where $\mu_i(x) = -\lambda_i(x) / \lambda_{k+1}(x), i = \overline{1,k}$.

Considering the first k equalities in (6) as a system of linear algebraic equations with respect to unknowns $\mu_1(x), \ldots, \mu_k(x)$, and with the same nonzero determinant, the following is obtained:

$$\mu_i(x) = \frac{W_{y_1,\ldots,y_{i-1},y_{k+1},y_{i+1},\ldots,y_k}(x)}{W_{y_1,\ldots,y_k}(x)}, \ i = \overline{1,k},$$

from which, in particular, it follows that $\mu_i(x) \in C^1(\alpha_1, \beta_1)$.

Next, we proceed as follows. We differentiate both parts of the first equation from (6) and subtract the second equation from the resulting equality:

$0=\mu_1'(x)y_1(x)+\ldots+\mu_k'(x)y_k(x)$, $x \in (\alpha_1, \beta_1)$. The same is done with the second and third equation, the third and fourth equation, etc. As a result, the following homogeneous system of equations with respect to derivatives $\mu_1'(x)$, ..., $\mu_k'(x)$ is obtained:

$$\begin{cases} 0=\mu_1'(x)y_1(x)+\ldots+\mu_k'(x)y_k(x), \\ 0=\mu_1'(x)y_1'(x)+\ldots+\mu_k'(x)y_k'(x), \\ \ldots \\ 0=\mu_1'(x)y_1^{(k-1)}(x)+\ldots+\mu_k'(x)y_k^{(k-1)}(x), \end{cases}$$

the determinant of which is $W_{y_1,\ldots,y_k}(x)\neq 0$ again. Hence, it may be concluded that $\mu_1'(x)=\ldots=\mu_k'(x)=0$ $\forall x\in(\alpha_1,\beta_1)$, and therefore $\mu_1(x), \ldots, \mu_k(x)$ are constants. Thus, due to the first equality in (6), solutions $y_1, y_2, \ldots, y_{k+1}$ are linearly dependent on the interval (α_1, β_1) and hence on (a, b). If an arbitrary solution y_{k+2} is taken now, then all generalized Wrońskians of the linearly dependent system $y_1, \ldots, y_{k+1}, y_{k+2}$ would be identically zero on (a, b). The lemma is proved.

Now, the main result of the work can be easily established.

THEOREM 1. Let $y_1(x), y_2(x), \ldots, y_k(x)$, $k \le n-1$ be solutions to Eq. (2), and let $W_{y_1,\ldots,y_k}(x)=0$ $\forall x\in(\alpha,\beta)\subset(a,b)$. Then functions $y_1, y_2, \ldots, y_k$ are linearly dependent on the interval (a, b).

PROOF. Suppose that solutions $y_1, y_2, \ldots, y_k$ are linearly independent. We supplement the system $y_1, y_2, \ldots, y_k$ to the fundamental system of solutions $y_1, y_2, \ldots, y_k, y_{k+1}, \ldots, y_n$ of Eq. (2). Applying Lemma 1 sequentially, it may be concluded that $W_{y_1,\ldots,y_{k+1}}(x)\equiv 0$, $W_{y_1,\ldots,y_{k+2}}(x)\equiv 0$, $\ldots$, $W_{y_1,\ldots,y_n}(x)\equiv 0$ on (a, b). But the equality to zero of the Wrońskian of the system of n solutions even at one point means their linear dependence. The obtained contradiction proves the theorem.

SPECIAL CASES

The following result shows that in some cases for linear dependence of solutions $y_1, y_2, \ldots, y_k$, it is sufficient to zeroize the Wrońskian $W_{y_1,\ldots,y_k}(x)$ on the set having a limit point.

THEOREM 2. Let the set of zeros of the Wrońskian $W_{y_1,\ldots,y_k}(x)$ of solutions $y_1(x), y_2(x), \ldots, y_k(x)$, $k\le n-1$ to Eq. (2) have the limit point $x_0 \in (a, b)$. Let, further, one of the following conditions be satisfied: (a) the coefficients of Eq. (2) are analytic functions (in particular, constant values) on the interval (a, b); (b) $k = 1$; (c) $k = n - 1$, and the coefficients of Eq. (2) satisfy the following smoothness conditions:

$$a_0(x)\in C(a,b),\ a_l(x)\in C^{l-1}(a,b),\ l=1,2,\ldots,n-2,$$
$$a_{n-1}(x)\in C^{n-3}(a,b).$$

Then solutions $y_1, y_2, \ldots, y_k$ are linearly dependent on (a, b).

PROOF. If condition (a) is satisfied, all solutions to Eq. (2) are analytic functions on the interval (a, b) ([14], Ch. 1, § 6), and hence the Wrońskian $W_{y_1,\ldots,y_k}(x)$ is analytic on (a, b). By the uniqueness theorem for analytic functions ([15], Ch. I, § 5, p. 20), $W_{y_1,\ldots,y_k}(x)\equiv 0$ on the interval (a, b), and it remains to refer to Theorem 1.

Let condition (b) be satisfied, i.e., $k = 1$. The Wrońskian of one solution is the solution itself, so our statement is an obvious consequence of the uniqueness theorem of the solution to the Cauchy problem.

We shall finally consider case (c). It should be noted that the result of differentiation of any generalized Wrońskian $W^{\alpha_1,\ldots,\alpha_k}_{y_1,\ldots,y_k}$ is a linear combination of some set of generalized Wrońskians of the same solutions $y_1, y_2, \ldots, y_k$. Indeed, if $\alpha_k < n-1$ in our determinant, then

$$\frac{d}{dx}W^{\alpha_1,\alpha_2,\ldots,\alpha_k}_{y_1,y_2,\ldots,y_k}=W^{\alpha_1+1,\alpha_2,\ldots,\alpha_k}_{y_1,y_2,\ldots,y_k}+ + W^{\alpha_1,\alpha_2+1,\ldots,\alpha_k}_{y_1,y_2,\ldots,y_k}+\ldots+W^{\alpha_1,\alpha_2,\ldots,\alpha_k+1}_{y_1,y_2,\ldots,y_k}, \tag{7}$$

where some of the obtained determinants may be equal to zero due to the presence of coincident lines in them.

If $\alpha_k = n-1$, then the last line of the last determinant in (7) will contain the nth derivatives of functions $y_1, y_2, \ldots, y_k$. Replacing the nth derivative of each solution by a linear combination of lower order derivatives due to Eq. (2), the following is obtained:

$$W^{\alpha_1,\ldots,\alpha_{k-1},n}_{y_1,y_2,\ldots,y_k}=-a_{n-1}W^{\alpha_1,\ldots,\alpha_{k-1},n-1}_{y_1,y_2,\ldots,y_k}- - a_{n-2}W^{\alpha_1,\ldots,\alpha_{k-1},n-2}_{y_1,y_2,\ldots,y_k}-\ldots-a_0W^{\alpha_1,\ldots,\alpha_{k-1},0}_{y_1,y_2,\ldots,y_k}. \tag{8}$$

Each of the determinants in the right-hand side of (8) is either zero or coincides with one of the generalized Wrońskians of solutions $y_1, \ldots, y_k$.

We shall apply these observations to the case of $k = n - 1$. It should be noted that there are exactly n different generalized Wrońskians of solutions $y_1, \ldots, y_{n-1}$ to Eq. (2): the Wrońskian $W_{y_1,\ldots,y_{n-1}}=W^{0,1,\ldots,n-2}_{y_1,\ldots,y_{n-1}}$ itself, determinants of the $W^{0,1,\ldots,l-1,l+1,\ldots,n-1}_{y_1,\ldots,y_{n-1}}$ form, and, finally, $W^{1,2,\ldots,n-1}_{y_1,\ldots,y_{n-1}}$. We simplify the notations assuming $W_{y_1,\ldots,y_{n-1}}=W$, $W^{0,1,\ldots,l-1,l+1,\ldots,n-1}_{y_1,\ldots,y_{n-1}}=W_l$, and $W^{1,2,\ldots,n-1}_{y_1,\ldots,y_{n-1}}=W_0$. Differentiating the determinants W and W_l according to

Eqs. (7) and (8) and discarding the resulting determinants equal to zero, the following is obtained:

$$W' = W_{n-2}, \tag{9}$$

$$W_l' = W_{l-1} - a_{n-1}W_l - (-1)^{n-l} a_l W, \quad l = \overline{n-2, 1}. \tag{10}$$

We show that the Wrońskian derivatives W of order 2 to $n - 1$ are expressed through the generalized Wrońskians using the following formulas:

$$W^{(j)} = W_{n-j-1} + \sum_{l=n-j}^{n-2} \alpha_{jl}(x) W_l + \beta_j(x) W, \quad j = \overline{2, n-1}, \tag{11}$$

where functions $\alpha_{jl}(x), \beta_j(x) \in C^{n-j-1}(a,b)$ (we already have Eq. (9) as the expression for the first derivative).

Indeed, we obtain the formula for W'' by differentiating (9) and substituting W_{n-2}' according to (10): $W'' = W_{n-3} - a_{n-1}W_{n-2} - a_{n-2}W$, which corresponds to (11), and the coefficients at the determinants in the right-hand side are functions of the $C^{n-3}(a,b)$ class due to the condition of the theorem. Further, assuming that formula (11) is true for some j, $2 \le j \le n - 2$, then, differentiating both parts of it and using (10), the following is obtained:

$$W^{(j+1)} = W_{n-j-2} - a_{n-1}W_{n-j-1} - (-1)^{j+1} a_{n-j-1} W + \sum_{l=n-j}^{n-2} [\alpha_{jl}' W_l + \alpha_{jl}(W_{l-1} - a_{n-1}W_l - (-1)^{n-l} a_l W)] + \beta_j' W + \beta_j W_{n-2} = W_{n-j-2} + \sum_{l=n-j-1}^{n-2} \alpha_{j+1,l} W_l + \beta_{j+1} W,$$

where $\alpha_{j+1,n-j-1} = -a_{n-1} + \alpha_{j,n-j}$,
$\alpha_{j+1,l} = \alpha_{jl}' - \alpha_{jl} a_{n-1} + \alpha_{j,l+1}$,
$l = n-j, n-j+1, \ldots, n-3$ (for $j \ge 3$),
$\alpha_{j+1,n-2} = \alpha_{j,n-2}' - \alpha_{j,n-2} a_{n-1} + \beta_j$,
$\beta_{j+1} = (-1)^j a_{n-j-1} - \sum_{l=n-j}^{n-2} (-1)^{n-l} \alpha_{jl} a_l + \beta_j'$.

It can be easily seen that due to the inductive assumption and the conditions of the theorem concerning the smoothness of the coefficients of equation $\alpha_{j+1,l}, \beta_{j+1} \in C^{n-j-2}(a,b)$, and thus (11) is proved.

Now let $x = x_0$ in (9) and (11). The point x_0, being the limit point for zeros of function $W(x)$, is such due to Rolle's theorem, and also for zeros of its derivatives $W'(x), \ldots, W^{(n-1)}(x)$. Hence, due to continuity, $W(x_0) = W'(x_0) = \ldots = W^{(n-1)}(x_0) = 0$. We obtain a linear homogeneous system of algebraic equations with respect to the unknowns $W_{n-2}(x_0)$, $W_{n-3}(x_0)$, …, $W_0(x_0)$ with triangular determinant different from zero:

$$0 = W_{n-2}(x_0),$$
$$0 = W_{n-j-1}(x_0) + \sum_{l=n-j}^{n-2} \alpha_{jl}(x_0) W_l(x_0), \quad j = \overline{2, n-1},$$

from which $W_{n-2}(x_0) = W_{n-3}(x_0) = \ldots = W_0(x_0) = 0$.

Then we use the already familiar technique: assuming that solutions y_1, …, y_{n-1} are linearly independent, we add one more solution to them to obtain the fundamental system of solutions $y_1, \ldots, y_{n-1}, y_n$ and, decomposing determinant $W_{y_1,\ldots,y_n}$ by the last column, we obtain $W_{y_1,\ldots,y_n}(x_0) = 0$, which means linear dependence $y_1, \ldots, y_n$. The obtained contradiction proves the linear dependence of solutions y_1, …, y_{n-1}. The theorem is proved.

COROLLARY. Let $y_1(x), y_2(x), \ldots, y_k(x)$, $x \in (a, b)$ be linearly independent solutions to Eq. (2). Then their Wrońskian cannot be identically zero on any interval $(\alpha, \beta) \subset (a, b)$. If one of conditions (a), (b) or (c) of Theorem 2 is satisfied, then the set of zeros of the determinant $W(x)$ cannot have limit points on the interval (a, b), or, equivalently, $W(x)$ cannot have an infinite number of zeros on any interval $[\alpha, \beta] \subset (a, b)$.

CONCLUSIONS

It follows from the above results that, in cases where the number of solutions is less than the order of the equation, the Wrońskian of a system of solutions to a linear homogeneous equation can be used to check whether the system is linearly dependent or independent; the solutions are linearly dependent if and only if their Wrońskian is identically equal to zero, and independent if the determinant is different from zero in at least one point. In this case, as Theorems 1 and 2 show, the verification of the identical equality to zero of the Wrońskian over the entire definition domain can be replaced by the verification of its equality to zero on a significantly smaller set, which facilitates the practical application of the results obtained.

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