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RESEARCH ARTICLE

Combined approximation algorithms for interactive design of road routes in CAD

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Abstract

Objectives. The aim of the work is to create algorithms for approximating a sequence of points on a plane by arcs of clothoids and circles. Such a problem typically arises in the design of railroad and highway routes. The plan (projection onto a horizontal plane) of the road route is a curve (spline) consisting of a repeating bundle of elements "straight line + clothoid arc + circle arc + clothoid arc + …". Such a combination of elements provides continuity not only for the curve and its tangent, but also for the curvature. Since the number of spline elements is not known in advance, and their parameters are subject to restrictions, there is no mathematically consistent algorithm for this problem. The two-stage scheme for solving the problem is developed at RTU MIREA only for a spline with lines and circles (i.e., without clothoid elements). At the first stage, the scheme uses dynamic programming to determine the number of spline elements. At the second stage, the scheme optimizes parameters of the spline using nonlinear programming. This scheme has yet to be implemented for a spline with clothoids due to a significantly more complicated nature of this problem. Therefore, the design of route plans in existing computer aided design (CAD) systems is carried out in interactive mode using iterative selection of elements. In this regard, it makes sense to develop mathematically consistent algorithms for element-by-element approximation.

Methods. The problem of element-by-element approximation by a circle and a clothoid is formalized as a low-dimensional non-linear programming problem. The objective function is the sum of squared deviations from the original points. Since a clothoid can only be represented in Cartesian coordinates by power series, there are difficulties in calculating the derivatives of the objective function with respect to the desired parameters of the spline elements. The proposed mathematically consistent algorithm for calculating these derivatives is based on the integral representation of the Cartesian coordinates of the points of the clothoid as functions of its length.

Results. A mathematical model and algorithms have been developed for approximating a sequence of points on a plane by clothoids and circles using the method of nonlinear programming. A second-order algorithm is implemented with the calculation and inversion of the matrix of second derivatives (Hesse matrix).

Conclusions. For approximation by circles and clothoids using nonlinear programming, it is not necessary to have an analytical expression of the objective function in terms of the required variables. The proposed algorithms make it possible to calculate not only the first, but also the second derivatives in the absence of such expressions.

Keywords: route plan, spline, non-linear programming, clothoid, objective function, gradient, Hessian matrix

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НАУЧНАЯ СТАТЬЯ

Комбинированные алгоритмы аппроксимации для интерактивного проектирования дорожных трасс в системах автоматизированного проектирования

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Резюме

Цели. Цель работы состоит в создании алгоритмов аппроксимации последовательности точек на плоскости дугами клотоид и окружностей. Такая задача возникает в проектировании трасс железных и автомобильных дорог. План (проекция на горизонтальную плоскость) трассы дороги – это кривая (сплайн), состоящая из повторяющейся связки элементов «прямая + дуга клотоиды + дуга окружность + дуга клотоиды + …». Такая комбинация элементов обеспечивает непрерывность не только кривой и касательной к ней, но и кривизны. Поскольку число элементов сплайна заранее неизвестно, а на их параметры накладываются ограничения, для этой задачи пока не опубликовано математически корректного алгоритма. Разработанная в РТУ МИРЭА двухэтапная схема решения задачи с определением числа элементов сплайна с помощью динамического программирования на первом этапе и оптимизацией его параметров с применением нелинейного программирования на втором, реализована только для сплайна с прямыми и окружностями (без клотоид). Ее реализация для сплайна с клотоидами много сложнее и пока не выполнена в силу ряда причин. В действующих системах автоматизированного проектирования (САПР) проектирование плана трассы выполняется в интерактивном режиме с последовательным подбором элементов. В этой связи имеет смысл разработка математически корректных алгоритмов поэлементной аппроксимации.

Метод. Задача поэлементной аппроксимации окружностью или клотоидой формализована как задача нелинейного программирования малой размерности. Целевая функция – сумма квадратов отклонений от исходных точек. Поскольку клотоида в декартовых координатах представляется степенными рядами, возникают трудности вычисления производных целевой функции по искомым параметрам элементов сплайна. Предложен математически корректный алгоритм вычисления этих производных на основе интегрального представления декартовых координат точек клотоиды как функций ее длины.

Результаты. Разработаны математическая модель и алгоритмы аппроксимации последовательности точек на плоскости клотоидой и окружностью с применением метода нелинейного программирования. Реализован алгоритм второго порядка с вычислением и обращением матрицы вторых производных (матрица Гессе).

Выводы. Для аппроксимации окружностью и клотоидой с применением нелинейного программирования необязательно иметь аналитическое выражение целевой функции через искомые переменные. Предложенные алгоритмы позволяют вычислять не только первые, но и вторые производные в отсутствие таких выражений.

Ключевые слова: план трассы, сплайн, нелинейное программирование, клотоида, целевая функция, градиент, матрица Гессе

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INTRODUCTION

The problem of approximating a curve defined by a sequence of points in the plane by circles and clothoids is much more complicated than the widely-used linear or parabolic regression approach. In the case of search for an approximating circle in Cartesian coordinates, the problem is reduced to a nonlinear system of equations. In the case of a clothoid, it is impossible to obtain even this, since such a curve cannot be represented in the form $y = f(x)$. A mathematically consistent algorithm for approximation by a clothoid has not yet been found in the literature.

Instead, current CAD solutions either use the method of enumeration of variants assigned by the designer or approximation by the involute method. The latter was proposed in the pre-computer era [1] as a means of calculating the shifts of a route plan to bring it to a given design position. During reconstruction of a railroad route plan, a design position is unknown. When changing plan parameters such as lengths of transition curves (clothoid arcs) or radii of circular curves, the design position is set in one way or another by the designer. After that, the computer calculates all design parameters of the route plan and shifts of the existing route. The involute method was also used for approximate solution of approximation problem instead of its exact solution in Cartesian coordinates[1]. This method has been successfully used to design a route plan as a whole within the railway haul back in the 1980s on such unsuitable for creating CAD computers as Minsk 32 and EU 1033.

The avoidance of Cartesian coordinates seemed to be forced. At the same time, it was known that the error of the method due to the presence of large angles and small radii can be very significant. This is especially true for heavily dislocated curves and correspondingly large shifts or when designing reconstruction with significant changes in the route plan parameters.

The involute method was also used in the 1980s for optimizing the horizontal alignment of new railroads in stressed sections [2, 3]. Here, the route plan was represented as a broken line to find the number of elements and their approximate location, which was then transformed to the required shape using this method with subsequent optimization of the resulting spline parameters [3]. Despite the disadvantages of the involute method, its wide application is explained by the fact that in contrast to Cartesian coordinates, it uses single-valued functions having simple analytical expressions: a parabola of the second order instead of a circle and a cubic parabola in the place of a clothoid.

Various heuristic algorithms were proposed in the works of authors [4–12]. Initially, these were based on analyzing initial broken line characteristics obtained after connecting adjacent survey points by line segments, such as rotation angles in the vertices of the broken line and curvature graphs. Subsequently, the use of genetic algorithms became more common. In Russia, various programs are used in current CAD systems to facilitate element-by-element selection and evaluation of trace variants. This approach has been most successfully realized by the Topomatic[2] company in the *Robur* system, which uses an existing semiautomatic trace plan parameter selection method based on curvature graphs. Although there have been claims to have solved the problem of designing railroad track plan reconstruction, the substantiating algorithms have yet to be published in full. Among recently published heuristic algorithms, the ones presented in the works of Chinese professor Hao Pu and colleagues [13–15] should be noted.

It is hard to believe in the existence of an optimal solution for several circular and transitive

[1] *Methodical recommendations for the calculation of composite curves of the railroad route plan*. Moscow: All-Russian Research Institute of Transport Construction; 1985. 26 p. (in Russ.).

[2] Topomatic Robur product documentation "Path selection via curvature graph." http://help.topomatic.ru/v6/doku.php?id=rail:rail:tasks:selection_path:start. Accessed April 18, 2023 (in Russ.).

curves (clothoids) if there is no optimal solution algorithm for a single clothoid.

As with the general design of plan- and longitudinal road profiles, the problem under consideration is characterized by the lack of a known optimal solution for complex cases. In the absence of designer interest in the achievement of such a solution, anything plausible that the computer gives out can be called "optimal". This represents the fundamental difference between this problem and those associated with modeling geometric shapes of roads [16].

Nevertheless, the development of mathematically substantiated algorithms of approximation by a circle and a clothoid remains both theoretically and practically relevant, since the result can be useful not only in road design. The up-to-date level of computer technology allows solving this problem in a reasonable time on publicly available computers and without the use of palliative algorithms.

The aim of this study is to present mathematically substantiated algorithms for approximation by a circle and a clothoid in Cartesian coordinates by means of nonlinear programming algorithms that use the involute method only to obtain an initial approximation, followed by an optimization of the parameters used to determine the position of the circle or clothoid.

PROBLEM STATEMENT

For a given sequence of points in the plane find a clothoid (circle) such that the sum of squares of distances h_i ($i = 1, 2, \dots, n$) from the given points to the clothoid (circle) is minimal. The distances are calculated using the normal line from a given point to the clothoid (circle).

The initial point of the desired curve is given. The direction of the tangent to the desired curve at the initial point and the minimum and maximum radius of curvature of the desired curve can also be specified.

Here, the objective function is

$$F(\mathbf{h}) = \frac{1}{2}\sum_{1}^{n} h_i^2 \to \min. \qquad (1)$$

Here, $\mathbf{h}(h_1, h_2, \dots, h_n)$ is the vector of unknowns, while n is their number.

INVOLUTE METHOD

Unlike other curves (parabolas, circles, sinusoids, etc.), one cannot speak of an involute without specifying another curve (evolute) that generates the involute. Various definitions of involute can be found in the literature. In mathematics, an involute is a *curve* for which a given evolute is the locus of curvature centers [17]. Consequently, the normal line at each point of an involute is a tangent to the evolute. This has to be a tangent rather than a secant because there can be only one center of curvature at each point of a curve. In this context, an involute is a curve described by the end of a flexible, inextensible thread coiled from an evolute (e.g., a circle).

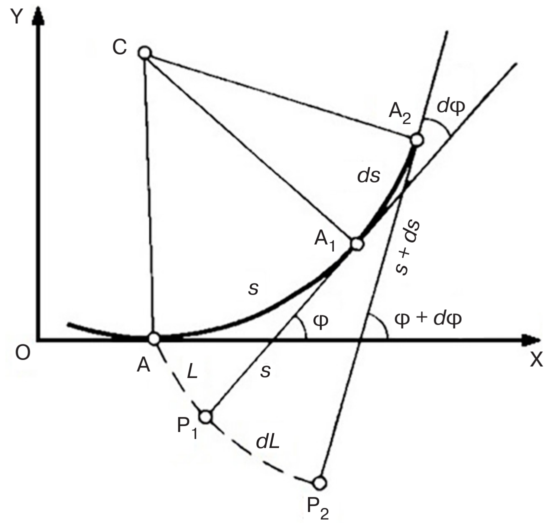

Fig. 1. Construction of the involute of a circle. *L* is the involute length

Figure 1 depicts the construction of the involute of a circle. Here, A is the initial point on the evolute, A_1 and A_2 are its new positions on the evolute. They correspond to the lengths of arcs counted from the initial point, s and $s + ds$. These lengths are tangentially set in the direction opposite to the motion from the initial point. The points P_1 and P_2 are obtained on the involute, respectively. The arc lengths of the involute are AP_1 and AP_2. At full rotation ($\varphi = 2\pi$) the length of the involute $E = 2\pi R$, where R is the radius of the circle. Thus, it is not the initial point A that is fixed, but its new positions on the evolute, in which tangents are constructed and on which the arcs of the involute are unfolded. *The involute of a circle is an unfolding spiral.*

However, in [1, 18, 19] a different treatment of the circle involute concept and the means of its construction is stated (Fig. 2). Here, the point A of the beginning of circular curve is fixed and for each point of the circle (a) the arc length from the initial point to the current one (Aa) is unfolded by a tangent from the point A. Thus, a *separate* involute is obtained (aa', bb' ...) for each point of the circle. Although the curve aa' is not a circle involute, its length is equal to the length of the corresponding section of involute. Hence, the confusion of the terms arose.

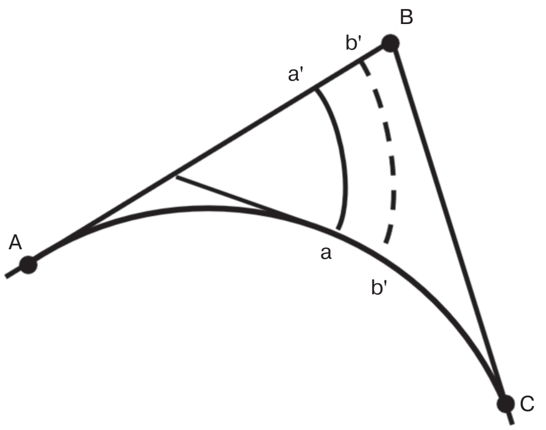

Fig. 2. Construction of the circle involute according to [1, 18, 19]

In [1, 18], circle involutes are *curves* constructed in a described way. But in [19, p. 243, Fig. 7.20], we can find: "*Involute* (italics of the authors) is the *length* of the arc aa', which will be described by the end of an inextensible thread, stretched on the circle Aa and fixed at the point A, when the thread is straightened."

If we assume that the involute is not the *curve* discussed above, but the *arc length* aa', then this length is $K^2/(2R)$, where K is the arc length Aa, R is the radius of the circle, while the circle involute is not a spiral in Cartesian coordinates, but a parabola of the second degree as a function of the evolute length.

If we follow the method of construction of the circle involute adopted in [1, 18, 19], then for each point of the circle (evolute) we will get a different involute, but they will all end on a tangent to the circle at the initial point, which is fixed. The true circle involute should be constructed as shown in Fig. 1.

From Fig. 1 it follows that $dL = \varphi(s)ds$ since $dL = sd\varphi = sds/R = \varphi ds$, and further:

$$L(S) = \int_0^S \varphi(s)ds. \quad (2)$$

Here S is the length of the evolute from the initial point, while $\varphi(s)$ is the dependence of the angle of the tangent to the evolute with the OX axis (or any given direction, which is not fundamental) on s, referred to as the angle diagram. The term "involute" was defined for smooth curves, and Eq. (2) is true for all such curves. However, in our problem (1), the initial curve is a broken line, and we cannot apply the above definitions of involute to it—i.e., consider the broken line as an involute. However, Eq. (2) can be generalized if we consider a curve whose length is calculated through the angle diagram of an evolute by Eq. (2), i.e., the length of an involute is the area of the angle diagram as a function of length of the original curve (particularly, a polyline). Thus, the involute itself is of no interest for the problem in question. Here, it is significant that the difference of lengths of involutes of two evolutes having a common point is, under some additional conditions, approximately equal to the distance between the evolutes along the normal. In our problem, the first evolute is a broken line connecting the approximated points, while the second is a design curve (a circle or a clothoid). Hence the involute method, which consists in the following:

1. We calculate the angles of the original broken line with OX axis and the values of the involute length L_i of the original broken line for survey points $i = 1, 2, \dots, n$.
   $L_1 = 0$, further

$$L_i = \sum_{j=2}^{i} \varphi_j s_j. \quad (3)$$

Here φ_j is the angle with OX axis of the jth leg of the broken line (from point $j - 1$ to point j), s_j is the length of this leg.

For a circle, the rotation angle depends linearly on the arc length, so the length of the design involute $L_{des}(S)$ is a second-order parabola.

2. If the design curve is given, the difference $L_{des}(S) - L_{init}(S)$ approximates the desired shift h_j, where j and S must match each other.
3. If we want to solve the approximation problem, the unknown coefficients of the desired parabola (a and b) are determined by the least square method.

For a circle, the involute *length* $L_{des}(S) = aS^2 + bS$, where S, as before, is the arc length from a given initial point. The meaning of the required parameters is as follows: $a = 1/(2R)$ and b is the angle with OX axis of the tangent to the circle at the initial point.

We obtain the problem:

$$F(a,b) = \sum_{i=2}^{n} \left(aS_i^2 + bS_i - L_i\right)^2 \to \min. \quad (4)$$

Here S_i is the length of the broken line from the first point to the ith survey point ($i = 2, \dots, n$), $S_1 = 0$.

Conditions $\frac{\partial F}{\partial a} = 0$ and $\frac{\partial F}{\partial b} = 0$ give a system of two linear equations, from which we obtain a and b; then $R = 1/(2a)$. Since the angle b can be given, only a remains unknown.

When approximating with a clothoid, the following options are possible: the clothoid connects a line to a circle; a circle to a line; two circles of various radii.

If the lengths of the contiguous clothoids are significantly shorter than the lengths of the circular curves, it is important to select the position of the circle, and only subsequently fit the clothoids. However, the opposite situation is also possible, where the clothoid is an independent element. For example, in highway design, one finds the term "clothoidal tracing" [20]. In this case, the route should not be thought of as consisting of only clothoids and straight lines, but clothoids prevail in the share by length.

By definition, the clothoid curvature $\sigma = \sigma_0 + kS$, where σ_0 is the curvature at the initial point (at length $S = 0$), k is the curvature change rate. The current angle of the tangent with OX axis is $\varphi = \varphi_0 + \sigma_0 S + kS^2/2$, where φ_0 is the initial angle of the tangent with OX axis. Finally, the length of the clothoid involute is equal to

$$L = \varphi_0 S + \sigma_0 S^2/2 + kS^3/6.$$

Clothoid approximation by the involute method in the general case is reduced to the problem:

$$F(\varphi_0, \sigma_0, k) = \\ = \sum_{i=2}^{n} \left(\varphi_0 S_i + \frac{\sigma_0 S_i^2}{2} + \frac{k S_i^3}{6} - L_i \right)^2 \to \min. \quad (5)$$

Here S_i and L_i have the same meaning as in the circle approximation problem (4). One or two variables can be fixed. In any case the problem (5) is solved simply. By differentiating (5) by the desired variables and equating the derivatives to zero, we obtain a system of linear equations (or one equation if two unknown variables are fixed). After solving the system, one should calculate the curvature at the end point of the clothoid $\sigma_n = \sigma_0 + kS_n$.

If $R_n = 1/\sigma_n$ appears outside the admissible limits, we should take $R_n = R_{\lim}$ and calculate $k = (1/R_{\lim} - \sigma_0)/S_n$ at a given σ_0. Then we obtain the unknown φ_0 by substituting the found k in (5) and solving the problem with one unknown variable. If σ_0 is not given, we substitute $\sigma_0 = \sigma_n - kS_n$ in (5) and solve the problem (5) with the remaining unknown variables and the found σ_n.

Thus, it is very easy to use the involute method for approximation by both a circle and clothoid. Unfortunately, the accuracy of the method may be insufficient. To test this statement, the Cartesian coordinates of the ends of the chords of a *given* length on a *given* circle were calculated as the coordinates of the initial survey points. Using these coordinates, the chord angles with the OX axis and involute lengths were calculated by Eq. (3). Then the problem (4) was solved. Obviously, its solution is the variables corresponding to the initial circle. In this case the deviations of all initial points from the approximating circle should be equal to zero. However, this was not the case. At circle radius $R = 500$ m or more, chords $l_x = 20$ m, and circle length $S \le 500$ m (i.e., at rotation angles less than 1 rad), the deviations of the obtained circle from the original were less than 0.01 m. However, at $R = 200$ m, $l_x = 20$ m, $S = 200$ m, the radius determined by the involute method was 199.9167 m instead of 200 m and the maximum deviation $D_{\max}$ was 0.0383 m instead of 0. At $S = 400$ m (rotation angle of 2 rad) for the same circle, $D_{\max} = 0.1180$ m instead of 0. In all cases, the length differences of design and initial involutes of the approximated circle in all points of survey were equal to zero. At $l_x = 10$ m, the accuracy of the method is significantly higher. Thus, at a division by 10 m we get $R = 299.9861$ and $D_{\max} = 0.00297$ if $R = 300$ m, $S = 200$ m, while at a division by 20 m, we obtain $R = 299.9444$ m and $D_{\max} = 0.01190$ m.

Similar calculations were performed with respect to the use of the involute method to approximate the clothoid. For compressing them, arcs of equal length were used, rather than chords, simplifying the calculation of Cartesian coordinates of points at the ends of arcs. These points were treated as survey points, chord lengths were calculated, and, as for circles, angles and involute lengths L_i were calculated using Eq. (3). Then, the problem (5) was solved with $\sigma_0 = 0$ and the unknown variables φ_0 and k.

As one would expect, the involute method works well for small values of the parameter k and short clothoids. Otherwise, the results are unsatisfactory.

Example 1. A clothoid of length 400 m is divided into 20 equal parts every 20.0 m; $k = 3.333333 \cdot 10^{-5}$. We obtain $\varphi_0 = -0.00027412$ instead of 0 and $k = 3.337930 \cdot 10^{-5}$ instead of the original $k = 3.333333 \cdot 10^{-5}$.

Approximation of the involute length as a function of length of a broken line by the cubic parabola (solution of problem 5) is performed with deviations not more than 0.041 m. However, the maximum deviation $D_{\max}$ of the initial points from the obtained clothoid along the normal line to it equal to 0.255 m.

Example 2. Same problem, but the length of the clothoid is 200 m. Approximation with the cubic parabola is performed with deviations not more than 0.001 m. We obtain $\varphi_0 = -1.839272551 \cdot 10^{-5}$ instead of 0 and $k = 3.334527 \cdot 10^{-5}$ instead of the original $k = 3.333333 \cdot 10^{-5}$, $D_{\max} = 0.048$ m.

It should be noted that these calculations show not so much how the involute method finds an optimum, but rather how it deviates from it. For a "heavily hit" initial route, the deviations from the optimum of the solutions obtained by the involute method may be significantly higher, especially for small radii and large rotation angles. However, in any case, these solutions can be used as initial approximations for the optimal approximation by a circle.

OPTIMIZATION OF THE INITIAL APPROXIMATION WHEN APPROXIMATING BY A CIRCLE

Let us assume that the coordinates of points to be approximated by the circle and its initial point A are given. Although we also usually fix the direction of the tangent at this point (angle α with OX axis), we will assume without loss of generality that this angle and the circle radius R are unknown. Coordinates of the circle center (Fig. 3) are

$$x_c = x_a - R \cdot \sin\alpha \text{ and } y_c = y_a + R \cdot \cos\alpha. \quad (6)$$

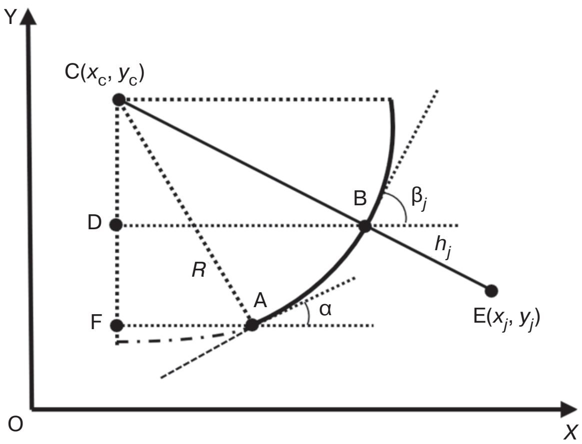

Fig. 3. Calculation of the derivatives of deviations h_j from the circle

Deviation h_j (BE in Fig. 3) of an arbitrary survey point E with coordinates x_j, y_j from the circle is

$$h_j = \sqrt{(x_j - x_c)^2 + (y_j - y_c)^2} - R. \quad (7)$$

Using (6) and (7), we obtain the derivatives:

$$\frac{\partial h_j}{\partial R} = \cos(\beta_j - \alpha) - 1 \text{ and } \frac{\partial h_j}{\partial \alpha} = R\sin(\beta_j - \alpha).$$

Here, β_j is the angle of the tangent to the circle at point B.

Then we calculate the gradient of the objective function (1):

$$\frac{\partial F}{\partial R} = \sum_{j=1}^{n} h_j \cdot \frac{\partial h_j}{\partial R} = \sum_{j=1}^{n} h_j(\cos(\beta_j - \alpha) - 1),$$
$$\frac{\partial F}{\partial \alpha} = \sum_{j=1}^{n} h_j \cdot \frac{\partial h_j}{\partial \alpha} = R\sum_{j=1}^{n} h_j \sin(\beta_j - \alpha). \quad (8)$$

The problem of approximation by a circle with objective function (1) is reduced to a two-dimensional minimization problem with restriction on R and initial approximation obtained by the method of involutes as a result of solving problem (4). Here, the presence of a good initial approximation is especially important, since there is no reason to believe that the problem is one-extremal.

Various methods can be used to solve the problem [21–23], in particular gradient or coordinate descent, changing alternately R and α.

Note, that if the initial point A and angle α are fixed, setting one more point of the circle (e.g., the end point) defines the circle uniquely. The enumeration of several such points and calculation of the value of the objective function for each of them may be sufficient, especially for large radii, when the involute method gives acceptable results, which can be improved and confirmed.

OPTIMIZATION OF THE INITIAL APPROXIMATION WHEN APPROXIMATING BY A CLOTHOID

As in the approximation by a circle, the main problem when approximating by clothoid is to calculate the derived distances of the given survey points from the clothoid by the parameters defining it. Let us show how this problem is solved by the example of the transition from a line to a circle using a clothoid.

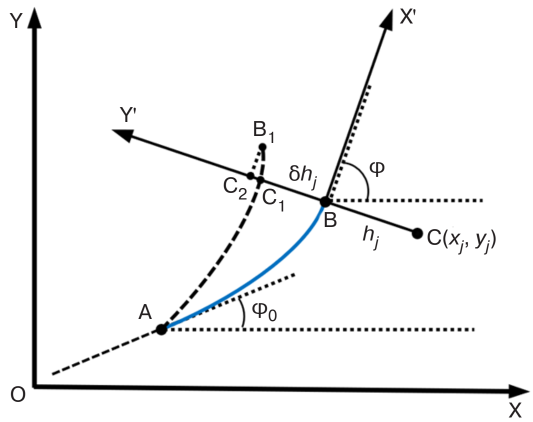

Fig. 4. To the calculation of the derivatives of the deviations h_j from the clothoid

In Fig. 4, AB is the initial position of the clothoid; AB_1 is its new position when one of the clothoid parameters is changed; φ_0 and φ are the angles of the tangent with the OX axis; CB = h_j is the deviation of the survey point C from the clothoid; BC_1 is its increment.

Let us denote the incremental coordinates of point B at its transition to a new position B_1 due to a change in any of the parameters of the clothoid by δx_B and δy_B. In the coordinate system with a center in point B and axes directed along the tangent and normal to the clothoid, respectively, the coordinate y' of the point B_1, i.e., BC_2 = = $\delta y_B \cos\varphi - \delta x_B \sin\varphi$. The value C_1C_2 can be neglected compared to C_1B, and the linear part of the increment can be assumed to be

$$\delta h_j = \delta y_B \cos\varphi - \delta x_B \sin\varphi. \quad (9)$$

Since the initial point A is fixed, the parameters determining the position of the clothoid—and hence the quality of the approximation—are in the general case: φ_0 is the angle of the tangent with the axis OX at the initial point (at $L = 0$), σ_0 is the initial curvature; k is the clothoid parameter (the curvature change rate, i.e., the curvature derivative over the length).

In the case of the transition from a line to a circle $\sigma_0 = 0$, only two variables φ_0 and k remain. In fact, the problem is reduced to the calculation of derived coordinates of points of the clothoid by these variables. Coordinates of intersection points of normals from each survey point with the clothoid and the length of the clothoid L_j from the initial point to the intersection points are calculated before the derivatives are calculated using the iterative algorithm along with the tangent angles with the OX axis [24].

Since the Cartesian coordinates of the clothoid points as functions of its lengths are expressed by power series, we will use their integral representation:

$$x_B = x_A + \int_0^L \cos\left(\varphi_0 + \frac{kt^2}{2}\right)dt,$$
$$y_B = y_A + \int_0^L \sin\left(\varphi_0 + \frac{kt^2}{2}\right)dt. \quad (10)$$

By differentiating (10), we obtain

$$\frac{\partial x_B}{\partial \varphi_0} = -\int_0^L \sin\left(\varphi_0 + \frac{kt^2}{2}\right)dt = -(y_B - y_A),$$

$$\frac{\partial y_B}{\partial \varphi_0} = \int_0^L \cos\left(\varphi_0 + \frac{kt^2}{2}\right)dt = (x_B - x_A),$$

$$\frac{\partial x_B}{\partial k} = -\int_0^L \sin\left(\varphi_0 + \frac{kt^2}{2}\right)\frac{t^2}{2}dt =$$
$$= \frac{1}{2k}\int_0^L td\cos\left(\varphi_0 + \frac{kt^2}{2}\right) = \frac{L\cos\varphi - (x_B - x_A)}{2k},$$

$$\frac{\partial y_B}{\partial k} = \int_0^L \cos\left(\varphi_0 + \frac{kt^2}{2}\right)\frac{t^2}{2}dt =$$
$$= \frac{1}{2k}\int_0^L td\sin\left(\varphi_0 + \frac{kt^2}{2}\right) = \frac{L\sin\varphi - (y_B - y_A)}{2k}.$$

Using (9), we obtain the required derivatives of the survey point deviations from the clothoid:

$$\frac{\partial h_j}{\partial \varphi_0} = (x_B - x_A)\cos\varphi + (y_B - y_A)\sin\varphi,$$
$$\frac{\partial h_j}{\partial k} = \frac{(x_B - x_A)\sin\varphi - (y_B - y_A)\cos\varphi}{2k}. \quad (11)$$

Here A is the initial point of the clothoid; h_j is the deviation of an arbitrary jth survey point from the clothoid along the normal line to it; B is the point where this normal line intersects the clothoid; φ is the angle of the tangent at this point with OX axis; L is the length of the clothoid from point A to point B.

Formulas (11) allow us to find the gradient of the function (1)

$$\frac{\partial F}{\partial k} = \sum_{j=1}^{n} h_j \cdot \frac{\partial h_j}{\partial k}; \quad \frac{\partial F}{\partial \varphi_0} = \sum_{j=1}^{n} h_j \cdot \frac{\partial h_j}{\partial \varphi_0}. \quad (12)$$

Further, it is possible to use gradient methods of optimization, e.g., the method of conjugate gradients [22–24]. However, Eqs. (10)–(12) allow a calculation of the matrix of second derivatives of the objective function **G** (Hessian matrix) to apply the more effective second-order method [23, 24]:

$$\frac{\partial^2 F}{\partial k^2} = \sum_{j=1}^{n}\left(\left(\frac{\partial h_j}{\partial k}\right)^2 + h_j \cdot \frac{\partial^2 h_j}{\partial k^2}\right) = G_{11},$$
$$\frac{\partial^2 F}{\partial k \partial \varphi_0} = \sum_{j=1}^{n}\left(\frac{\partial h_j}{\partial \varphi_0} \cdot \frac{\partial h_j}{\partial k} + \frac{\partial^2 h_j}{\partial k \partial \varphi_0}\right) = G_{12} = G_{21}, \quad (13)$$
$$\frac{\partial^2 F}{\partial \varphi_0^2} = \sum_{j=1}^{n}\left(\left(\frac{\partial h_j}{\partial \varphi_0}\right)^2 + h_j \cdot \frac{\partial^2 h_j}{\partial \varphi_0^2}\right) = G_{22}.$$

Calculation of the Hessian matrix is reduced to the calculation of the second derivatives of the deviations h_j of the approximated points from the clothoid by the desired variables φ_0 and k, since the first derivatives have already been calculated.

In accordance with (9)

$$\frac{\partial^2 h_j}{\partial k^2} = -\sin\varphi \cdot \frac{\partial^2 x_B}{\partial k^2} + \cos\varphi \cdot \frac{\partial^2 y_B}{\partial k^2} =$$
$$= \sin\varphi \int_0^L \cos\left(\varphi_0 + \frac{kt^2}{2}\right)\frac{t^4}{4}dt -$$
$$-\cos\varphi \int_0^L \sin\left(\varphi_0 + \frac{kt^2}{2}\right)\frac{t^4}{4}dt.$$

Integrating by parts and using the previously calculated integrals, we obtain

$$\frac{\partial^2 h_j}{\partial k^2} = \frac{1}{4k}\left\{L^3 + \frac{3}{k}(\cos\varphi(y_B - y_A) - \sin\varphi(x_B - x_A))\right\},$$

$$\frac{\partial^2 h_j}{\partial \varphi_0^2} = -\sin\varphi \cdot \frac{\partial^2 x_B}{\partial \varphi_0^2} + \cos\varphi \cdot \frac{\partial^2 y_B}{\partial \varphi_0^2} =$$
$$= \sin\varphi(x_B - x_A) - \cos\varphi(y_B - y_A),$$
$$\frac{\partial^2 h_j}{\partial k \partial \varphi_0} = \sin\varphi \int_0^L \cos\left(\varphi_0 + \frac{kt^2}{2}\right)\frac{t^2}{2}dt - \qquad (14)$$
$$-\cos\varphi \int_0^L \sin\left(\varphi_0 + \frac{kt^2}{2}\right)\frac{t^2}{2}dt =$$
$$= \frac{1}{2k}\{L - \sin\varphi(y_B - y_A) - \cos\varphi(x_B - x_A)\}.$$

Using the derivatives obtained, we calculate the Hessian matrix and its inverse $\mathbf{G}^{-1}$.

Let us denote: $\mathbf{x}(\varphi_0, k)$ is the vector of the unknown variables, $\mathbf{g}\left(\frac{\partial F}{\partial \varphi_0}, \frac{\partial F}{\partial k}\right)$ is the gradient. For the initial approximation obtained by the involute method (zero iteration) these are $\mathbf{x^0}$ and $\mathbf{g^0}$. Let us calculate the next iteration point:

$$x^1 = x^0 - G^{-1} \cdot g^0. \qquad (15)$$

For a positively defined Hessian matrix, this is the point of minimum of the objective function when decomposed into a Taylor series and restricted to the second derivatives. In the general case, one step to the point of minimum on the ray $\mathbf{x}^0 - \lambda\mathbf{G}^{-1} \cdot \mathbf{g}^0$ is insufficient. In the quadratic problem, the minimum is reached at $\lambda = 1$. In general case at $\lambda = 1$, we obtain the minimum point not of the original function, but of the approximating quadratic form with the Hessian matrix, which is not the same thing. Therefore, $\lambda = 1$ should be regarded only as an approximate value: the exact value should be sought by solving the problem of one-dimensional optimization of the function $F(\mathbf{x})$ on the ray $\mathbf{x}^0 - \lambda\mathbf{G}^{-1} \cdot \mathbf{g}^0$, considering it as a function of the single parameter λ.

At the resulting point of minimum in the direction, we again have to calculate the gradient and the matrix inverse to the Hessian matrix, and so on.

In general, the algorithm of the clothoid approximation consists of the following steps.

1. Construct the angular diagram of the original broken line by survey points.
2. Calculate the involute length of the broken line sequentially by the survey points by the formula (3).
3. Solve the problem of approximation of the obtained broken line. In general case, approximation by cubic parabola in the presence of three unknown parameters of clothoid or by square parabola in the presence of two parameters, as considered above. In this case, the system of no more than three linear equations is solved.
4. For a clothoid corresponding to the obtained solution, a special iterative algorithm determines the intersection points with the normals from each survey point, the angles of the tangents at these points with the OX axis, the corresponding lengths from the initial point of the clothoid to each of them, and the deviations h_j of the survey points from the clothoid.
5. The first and second derivatives of the deviations h_j are calculated for the required parameters of the clothoid (12), (13).
6. The gradient of the objective function (14) is calculated and the conditions of the end of counting are checked (for example, the smallness of the gradient norm). If the count terminating conditions are not satisfied, then:
7. Hessian matrix and its inverse are calculated.
8. The point of minimum of corresponding quadratic form (15) is determined, the problem of one-dimensional optimization, i.e., correction of step in search direction, is solved, and transition to new iteration point and further with new values of unknowns to item 4 is performed.

To adjust and verify the clothoid approximation algorithm, we first used the results of the test problems using the involute method, for which the optimum was known, but the involute solutions were unsatisfactory.

In the involute method Example 1 discussed above, the maximum deviation D_{max} = 0.255 m instead of 0, $\varphi_0 = -0.00027412$ instead of 0 and $k = 3.337930 \cdot 10^{-5}$ instead of $k = 3.333333 \cdot 10^{-5}$. Objective function $F^0 = 0.10298$. At the point of minimum of the quadratic form (15), $F = 1.08524 \cdot 10^{-7}$, $\varphi_0 = 3.099241 \cdot 10^{-7}$, $k = 3.333335 \cdot 10^{-5}$. After the first iteration $F = 3.70261 \cdot 10^{-15}$, $\varphi_0 = -2.422261 \cdot 10^{-14}$, $k = 3.333333333 \cdot 10^{-5}$. No deviations exceeded 0.00006 m.

In Example 2, the same problem was solved, but with a clothoid length of 200 m. By the involute method, $\varphi_0 = -1.839272551 \cdot 10^{-5}$ instead of 0 and $k = 3.334527 \cdot 10^{-5}$ instead of $k = 3.333333 \cdot 10^{-5}$ were obtained. D_{max} = 0.048 m. At the minimum point of the quadratic form (15) $F = 1.80438 \cdot 10^{-14}$, $\varphi_0 = -9.592883457 \cdot 10^{-10}$, $k = 3.333334 \cdot 10^{-5}$, D_{max} = 0.0002 m. The iteration was interrupted due to achieving the required gradient accuracy.

CONCLUSIONS

New possibilities offered by contemporary publicly available computers, along with the theory and methods of computer development of design solutions, are far from being fully utilized in existing linear structure CAD solutions, which are based on ideas from more than 50 years ago. Transition to the development and introduction of intelligent CAD, in which design solutions are given by computer as a result of optimization problem solving, is already possible in many design problems. However, because of lack of consumer interest in creating such systems and the high labor and funds for their development in Russia, such transition is unlikely in the near future. The most advanced in this respect are the mentioned works of Chinese scientists. However, existing CAD systems can be improved by applying optimization programs in the interactive process of making design decisions. Thus, the approximation algorithms outlined in the article can be applied instead of manually assigned solutions. The outlined clothoid approximation algorithm may be useful also when solving problems not related to roadway design.

The closest task is the generalization of the outlined method for calculating the derivatives of the objective function by the parameters determining the curve as a whole (rather than a single clothoid), in the absence of analytical expression of this function through the required parameters. We are talking about the transition from the solution of the problem of approximation by a single clothoid to spline approximation by the sequence of several bundles of "straight line + clothoid + circle + clothoid +...".

Authors' contribution. All authors equally contributed to the present work.

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