Mathematical modeling

Математическое моделирование

UDC 539.3
https://doi.org/10.32362/2500-316X-2023-11-3-70-85

RESEARCH ARTICLE

Developing generalized model representations of thermal shock for local non-equilibrium heat transfer processes

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Abstract
Objectives. Processes of energy transfer in solids and resultant thermal loads are widespread in nature and technology. This explains the scientific and practical significance of constructing a theory of these processes, as well as developing effective methods for studying the modeled concepts developed on this basis. The purpose of such studies is to determine basic flux patterns of complex processes occurring especially under conditions of powerful energy impacts in various technological operations. These include plasma-chemical processing of materials, their processing in infrared furnaces and solar plants, intense heating of materials carried out by laser or electron beams, and the use of powerful radiation emitters for thermal hardening and hardening of the surface of products. In these cases, the phenomenon of thermal shock arises, forming one of the central topics in thermomechanics and strength physics of solids. The present work considers an open theoretical problem of thermal shock in terms of a generalized model of dynamic thermoelasticity under conditions of a locally nonequilibrium heat transfer process. Depending on the type and curvature of the boundary surface of the considered massive body, the model can be used to study the problem in three coordinate systems: cartesian coordinates—a massive body bounded by a flat surface; spherical coordinates—a massive body with an internal spherical cavity; cylindrical coordinates—a massive body with an internal cylindrical cavity. Three types of intensive heating are considered: temperature heating, thermal heating, and heating by medium. Following the development of an analytical solution, the results of conducted numerical experiments are presented along with their physical analysis.
Methods. The study applies methods and theorems of operational calculus according to the theory of special functions.
Results. Generalized model representations of thermal shock are developed in terms of dynamic thermoelasticity for locally nonequilibrium heat transfer processes simultaneously in three coordinate systems: Cartesian, spherical, and cylindrical. The presence of curvature of the boundary surface of the thermal shock area substantiates the initial statement of the dynamic problem in displacements using the proposed corresponding "compatibility" equation.
Conclusions. A generalized dynamic model of the thermal reaction of massive bodies with internal cavities simultaneously in Cartesian, spherical, and cylindrical coordinate systems under conditions of intense temperature heating, thermal heating, and heating by medium is proposed. The model is considered in terms of displacements based on local nonequilibrium heat transfer. A numerical experiment carried out according to the obtained analytical solution for stresses forms a basis for a description of the wave nature of the propagation of a thermoelastic wave. A comparison with the classical solution is made without taking into account local nonequilibrium. The calculation of engineering relations carried out on the basis of the operational solution of the problem is important in practical terms for the upper estimate of the maximum thermal stresses.

Keywords: heat stroke, generalized dynamic model, analytical solution, thermal stresses

• **Submitted:** 24.06.2022 • **Revised:** 16.12.2022 • **Accepted:** 03.03.2023

For citation: Kartashov E.M. Developing generalized model representations of thermal shock for local non-equilibrium heat transfer processes. *Russ. Technol. J*. 2023;11(3):70–85. https://doi.org/10.32362/2500-316X-2023-11-3-70-85

Financial disclosure: The author has no a financial or property interest in any material or method mentioned.

The author declares no conflicts of interest.

НАУЧНАЯ СТАТЬЯ

Развитие обобщенных модельных представлений теплового удара для локально-неравновесных процессов переноса теплоты

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Резюме
Цели. Процессы переноса энергии в твердых телах и вызываемые ими тепловые нагрузки имеют широкое распространение в природе и технике. Этим объясняется исключительно важное научное и практическое значение построения теории указанных процессов, создание эффективных методов исследования развиваемых при этом модельных представлений. Цель этих исследований – установление основных закономерностей протекания достаточно сложных процессов, особенно в условиях мощных энергетических воздействий в различного рода технологических операциях. К ним можно отнести плазмохимическую обработку материалов, обработку в инфракрасных печах и гелиоустановках, интенсивный нагрев материалов лазерными или электронными лучами, применение мощных радиационных излучателей для термической закалки и упрочнения поверхности изделий. В этих случаях возникает так называемый термический удар – одна из центральных тем в термомеханике и физике прочности твердых тел, имеющая важное научное и практическое значение. Цель работы – рассмотреть открытую проблему теории теплового удара в терминах обобщенной модели динамической термоупругости в условиях локально-неравновесного процесса переноса теплоты. Модель (в зависимости от вида и кривизны граничной поверхности рассматриваемого массивного тела) позволяет исследовать проблему в трех системах координат: декартовы координаты – массивное тело, ограниченное плоской поверхностью; сферические координаты – массивное тело с внутренней сферической полостью; цилиндрические координаты – массивное тело с внутренней цилиндрической полостью. Рассматриваются три вида интенсивного нагрева: температурный, тепловой, нагрев средой. Ставится задача: получить аналитическое решение, провести численные эксперименты и дать их физический анализ.
Методы. Использованы методы и теоремы операционного исчисления, теория специальных функций.
Результаты. Развиты обобщенные модельные представления теплового удара в терминах динамической термоупругости для локально-неравновесных процессов переноса теплоты одновременно в трех системах координат: декартовой, сферической и цилиндрической. Наличие кривизны граничной поверхности области теплового удара обосновывает исходную постановку динамической задачи в перемещениях с использованием предложенного соответствующего уравнения «совместности».
Выводы. Предложена обобщенная динамическая модель термической реакции массивных тел с внутренними полостями одновременно в декартовой, сферической и цилиндрической системах координат в условиях интенсивного температурного нагрева, теплового нагрева, нагрева средой. Модель рассмотрена в перемещениях на основе локально-неравновесного теплопереноса. Получено аналитическое решение для напряжений, проведен численный эксперимент; описан волновой характер распространения термоупругой волны.

Проведено сравнение с классическим решением без учета локальной неравновесности. На основе операционного решения задачи предложены важные в практическом отношении расчетные инженерные соотношения для верхней оценки максимума термических напряжений.

Ключевые слова: тепловой удар, обобщенная динамическая модель, аналитическое решение, термические напряжения

• **Поступила:** 24.06.2022 • **Доработана:** 16.12.2022 • **Принята к опубликованию:** 03.03.2023

Для цитирования: Карташов Э.М. Развитие обобщенных модельных представлений теплового удара для локально-неравновесных процессов переноса теплоты. *Russ. Technol. J.* 2023;11(3):70−85. https://doi.org/10.32362/2500-316X-2023-11-3-70-85

Прозрачность финансовой деятельности: Автор не имеет финансовой заинтересованности в представленных материалах или методах.

Автор заявляет об отсутствии конфликта интересов.

INTRODUCTION

The present work continues the author's earlier research described in [1, 2]. Within the generalized model of dynamic thermoelasticity, the open problem of the thermal response to heating of a massive body bounded from the inside either by a flat surface (an elastic half-space in the Cartesian coordinate system), or a cylindrical surface (an elastic space in the cylindrical coordinate system with an inner cylindrical cavity), or a spherical surface (an elastic space in the spherical coordinate system with an inner spherical cavity), is studied. Three cases of intense heating of the body (region) boundary are considered: temperature heating, thermal heating, and heating by medium.

The development of generalized models involving numerous practical applications is one of the insufficiently studied directions in thermal shock theory. The specificity of such research lies, on the one hand, in the relative simplicity of initial mathematical models, and on the other, in computational difficulties in obtaining the desired result. Moreover, the obtained relations have obvious relevance for solving numerous practical situations.

The study is conducted under the conditions of locally nonequilibrium heat transfer processes [3–8]. Two circumstances are considered. During high-intensity heating of solids causing thermal shock, heat fluxes $\bar{q}(M,t)$,—where t is time in region $\Omega = \{M(x,y,z) \in D, t>0\}$ describing a real solid body—lag behind the temperature gradient $T(M, t)$ by a value proportional to relaxation time τ_r related to the heat propagation rate v_T by relation $v_T = \sqrt{a/\tau_r}$; here, a is temperature conductivity [9–15]:

$$\bar{q}(M,t) = -\lambda_T \operatorname{grad} T(M,t) - \tau_r \frac{\partial \bar{q}(M,t)}{\partial t}, \qquad (1)$$

where λ_T is the thermal conductivity.

The combination of the energy equation for isotropic solids $c\rho^* \partial T(M,t)/\partial t = \operatorname{div} \bar{q}(M,t)$, where c is specific heat capacity while ρ^* is the density, and Eq. (1) result in the following hyperbolic transfer equation [16, 17]:

$$\frac{\partial T(M,t)}{\partial t} = a\Delta T(M,t) - \tau_r \frac{\partial^2 T(M,t)}{\partial t^2}, \qquad (2)$$

containing not only the first, but also the second derivative of the temperature over time. As a consequence, Eq. (2) describes wave processes—in this case, wave heat transfer. While issues of correct formulation of boundary value problems for Eq. (2) have been considered relatively recently [16], a number of issues related to associated thermal problems require further study.

DEFINING RELATIONS OF DYNAMIC THERMOELASTICITY

Let D be the finite or partially bounded convex region of space $M(x, y, z)$ describing the real solid body and being in the thermally stressed state; S be the piecewise smooth surface bounding region D; $\bar{n} = (n_1, n_2, n_3)$ be the external normal to S (the vector continuous on S); $T(M, t)$ be the temperature distribution in region D at $t > 0$; T_0 be the initial temperature at which the region is in the unstrained and unstressed state. Let $\sigma_{ij}(M, t)$, $\varepsilon_{ij}(M, t)$, and $U_i(M, t)$ be the components of the stress tensor, strain tensor, and displacement vector, respectively, satisfying basic equations of (uncoupled) thermoelasticity (in index notation) [17]:

$$\sigma_{ij,j}(M, t) + F_i(M, t) = \rho^* \ddot{U}_i(M, t), \qquad (3)$$

$$\varepsilon_{ij}(M, t) = (1/2)[U_{i,j}(M, t) + U_{j,i}(M, t)], \qquad (4)$$

$$\sigma_{ij}(M, t) = 2\mu\varepsilon_{ij}(M, t) + [\lambda\varepsilon_{ii}(M, t) - (3\lambda + 2\mu)\alpha_T(T(M, t) - T_0)]\delta_{ij}, \; M \in D, \; t > 0, \qquad (5)$$

where ρ^* is density; λ, μ are isothermal Lame coefficients; G is the shear modulus; $\lambda = 2G\nu/(1-2\nu)$; ν is the Poisson's ratio, with $2G(1+\nu) = E$, where E is the Young's modulus; α_T is the linear thermal expansion coefficient; δ_{ij} is the Kronecker symbol; $F_i(M, t)$ are components of volume force; $\overline{e}(M,t) = U_{i,i}(M,t) = \varepsilon_{ii}(M,t)$ is the volume strain, which is related to the sum of normal stresses $\overline{\sigma}(M,t) = \sigma_{nn}(M,t),\ n = x, y, z$ by the following relation:

$$\overline{e}(M,t) = \frac{1-2\nu}{E}\overline{\sigma}(M,t) + 3\alpha_T\left[T(M,t) - T_0\right]. \quad (6)$$

The thermal stress state of region D at $t > 0$ may arise under various modes of thermal effect on boundary S, creating a thermal shock. These may include the most common in practice cases:

- temperature heating

$$T(M, t) = T_c(t),\ M \in S,\ t > 0\ (T_c(t) > T_0,\ t \geq 0); \quad (7)$$

- thermal heating

$$\frac{1}{\tau_r}\int_0^t \frac{\partial T(M,\tau)}{\partial n}\Big|_{M\in S} \exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau = -\frac{1}{\lambda_T}q(t)S_+(t),\ t \geq 0; \quad (8)$$

- heating by medium

$$\frac{1}{\tau_r}\int_0^t \frac{\partial T(M,\tau)}{\partial n}\Big|_{M\in S} \exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau = h\left\{T(M,t)\big|_{M\in S} - \left[T_0 + S_+(t)(T_c - T_0)\right]\right\},\ t \geq 0, \quad (9)$$

as well as under the action of internal heat sources. Here, λ_T is the thermal conductivity of the material; $q(t)$ is the heat flux; h is the relative heat transfer coefficient; T_c is the ambient temperature; $S_+(t) = \begin{cases} 1,\ t > 0, \\ 0,\ t \leq 0. \end{cases}$

Also, the cases of solid cooling can be equally considered.

The temperature function $T(M, t)$ included in (5) is found from the solution of the boundary-value problem of unsteady thermal conductivity for Eq. (2) with boundary conditions (7)–(9). Relations (3)–(6) are general relations of dynamic thermoelasticity linking stresses, strains, displacements, and temperature. When passing to specific cases, it is necessary to transform (3)–(6) into the so-called compatibility equations either in stresses or in displacements and to write the corresponding problem of dynamic thermoelasticity for these equations. For the case considered in the paper, it is necessary to take into account the effect of the solid boundary surface curvature on the temperature and the corresponding temperature stresses. Here, the more convenient mathematical model is the "compatibility" equation in displacements covering the cylindrical, spherical, and Cartesian coordinate systems, simultaneously.

Substituting the right parts of (5) into (3) (without volume forces) and using further (4) and (6), the following three equations may be written after a number of long transformations:

$$\Delta U_i(M,t) + \frac{1}{(1-2\nu)}\cdot\frac{\partial\overline{e}(M,t)}{\partial i} - (\rho^*/G)\frac{\partial^2 U_i(M,t)}{\partial t^2} = \frac{2(1+\nu)\alpha_T}{(1-2\nu)}\cdot\frac{\partial\left[T(M,t) - T_0\right]}{\partial i},\ i = x, y, z,$$

which formally can be written in the following form of vector equality:

$$\Delta\overline{U}(M,t) + \frac{1}{(1-2\nu)}\operatorname{grad}\left[div\overline{U}(M,t)\right] - (\rho^*/G)\frac{\partial^2\overline{U}(M,t)}{\partial t^2} = \frac{2(1+\nu)}{(1-2\nu)}\alpha_T\operatorname{grad}\left[T(M,t) - T_0\right],\ M \in D,\ t > 0. \quad (10)$$

It should be noted that it is necessary to equate the corresponding components in the vector record of the left and right parts in (10) in the inverse transition.

Let us further consider practical cases of dynamic thermoelasticity based on Eq. (10). In the first case, in Cartesian coordinates (x, y, z), region $z > R$, $t > 0$ bounded by the flat surface whose temperature state is described by function $T_i(z, t)$, $i = 1, 2, 3$ is considered; thus, $U_x = U_y = 0$, $U_z = U_z(z, t)$, and Eq. (10) may be written as follows:

$$\frac{\partial^2 U_z(z,t)}{\partial z^2} - \frac{1}{v_p^2}\cdot\frac{\partial^2 U_z(z,t)}{\partial t^2} = \frac{1+\nu}{1-\nu}\cdot\alpha_T\frac{\partial[T_i(z,t) - T_0]}{\partial z},\ z > R,\ t > 0. \quad (11)$$

Here, $v_p = \sqrt{\dfrac{2G(1-\nu)}{\rho^*(1-2\nu)}} = \sqrt{(\lambda + 2\mu)/\rho^*}$ is the speed of expansion wave propagation in elastic medium, which is close to the speed of sound.

The desired stress component $\sigma_{zz}(z, t)$ is related to displacement by the following relation:

$$\sigma_{zz}(z,t) = 2G\left\{\frac{1-\nu}{1-2\nu}\cdot\frac{\partial U_z(z,t)}{\partial z} - \frac{1+\nu}{1-2\nu}\cdot\alpha_T\left[T_i(z,t) - T_0\right]\right\}. \quad (12)$$

The temperature function satisfies three heating conditions

$$\frac{\partial T_i(z,t)}{\partial t}=a\frac{\partial^2 T_i(z,t)}{\partial z^2}-\tau_r\frac{\partial^2 T_i(z,t)}{\partial t^2},\quad z>R,\ t>0, \tag{13}$$

$$T_i(z,t)\big|_{t=0}=T_0,\ \left.\frac{\partial T_i(z,t)}{\partial t}\right|_{t=0}=0,\quad z\ge R, \left|T_i(z,t)\right|<\infty,\ z\ge R,\ t\ge 0, \tag{14}$$

$$T_i(z,t)\big|_{z=R}=T_c,\ t>0, \tag{15}$$

$$\frac{1}{\tau_r}\int_0^t \left.\frac{\partial T_2(z,\tau)}{\partial z}\right|_{z=R}\exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau=-\frac{1}{\lambda_T}q_0,\ t>0, \tag{16}$$

$$\frac{1}{\tau_r}\int_0^t \left.\frac{\partial T_3(z,\tau)}{\partial z}\right|_{z=R}\exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau= =h\left[T_3(z,t)\big|_{z=R}-T_c\right],\ t>0. \tag{17}$$

In the second case, in spherical coordinates (ρ, φ, θ), region $\rho>R$, $t>0$ with inner spherical cavity when heated under conditions of central symmetry $T_i=T_i(\rho,t)$ is considered; thus, $U_\varphi=U_\theta=0$, $U_\rho=U_\rho(\rho,t)$, and (12) may be written in the following form:

$$\frac{\partial U_\rho(\rho,t)}{\partial \rho^2}+\frac{2}{\rho}\cdot\frac{\partial U_\rho(\rho,t)}{\partial\rho}- -\frac{2}{\rho^2}\cdot U_\rho(\rho,t)-\frac{1}{v_p^2}\cdot\frac{\partial^2 U_\rho(\rho,t)}{\partial t^2}= =\frac{1+\nu}{1-\nu}\cdot\alpha_T\cdot\frac{\partial[T_i(\rho,t)-T_0]}{\partial\rho},\ \rho>R,\ t>0. \tag{18}$$

Here,

$$\sigma_{\rho\rho}(\rho,t)=2G\left\{\frac{1-\nu}{1-2\nu}\cdot\frac{\partial U_\rho(\rho,t)}{\partial\rho}+\frac{2\nu}{1-2\nu}\cdot\frac{1}{\rho}\times \times U_\rho(\rho,t)-\frac{1+\nu}{1-2\nu}\cdot\alpha_T\left[T_i(\rho,t)-T_0\right]\right\}, \tag{19}$$

$$\frac{\partial T_i(\rho,t)}{\partial t}=a\left(\frac{\partial^2 T_i(\rho,t)}{\partial\rho^2}+\frac{2}{\rho}\cdot\frac{\partial T_i(\rho,t)}{\partial\rho}\right)- -\tau_r\cdot\frac{\partial^2 T_i(\rho,t)}{\partial t^2},\ \rho>R,\ t>0, \tag{20}$$

$$T_i(\rho,t)\big|_{t=0}=T_0,\ \left.\frac{\partial T_i(\rho,t)}{\partial t}\right|_{t=0}=0,\quad \rho\ge R,\ \left|T_i(\rho,t)\right|<\infty,\ \rho\ge R,\ t\ge 0, \tag{21}$$

$$T_i(\rho,t)\big|_{\rho=R}=T_c,\ t>0, \tag{22}$$

$$\frac{1}{\tau_r}\int_0^t \left.\frac{\partial T_2(\rho,\tau)}{\partial\rho}\right|_{\rho=R}\exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau=-\frac{1}{\lambda_T}q_0,\ t>0, \tag{23}$$

$$\frac{1}{\tau_r}\int_0^t \left.\frac{\partial T_3(\rho,\tau)}{\partial\rho}\right|_{\rho=R}\exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau= =h\left[T_3(\rho,t)\big|_{\rho=R}-T_c\right],\ t>0. \tag{24}$$

In the third case, in cylindrical coordinates (r, φ, z), region $r>R$, $t>0$ with inner cylindrical cavity under radial temperature conditions $T_i=T_i(r,t)$ is considered; thus, $U_\varphi=U_z=0$, $U_r=U_r(r,t)$, and Eq. (10) may be written in the following form:

$$\frac{\partial^2 U_r(r,t)}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial U_r(r,t)}{\partial r}- -\frac{1}{r^2}\cdot U_r(r,t)-\frac{1}{v_p^2}\cdot\frac{\partial^2 U_r(r,t)}{\partial t^2}= =\frac{1+\nu}{1-\nu}\cdot\alpha_T\cdot\frac{\partial\left[T_i(r,t)-T_0\right]}{\partial r},\ r>R,\ t>0. \tag{25}$$

Then,

$$\sigma_{rr}(r,t)=2G\left\{\frac{1-\nu}{1-2\nu}\cdot\frac{\partial U_r(r,t)}{\partial r}+\frac{\nu}{1-2\nu}\times \times\frac{1}{r}U_r(r,t)-\frac{1+\nu}{1-2\nu}\cdot\alpha_T\left[T_i(r,t)-T_0\right]\right\}, \tag{26}$$

$$\frac{\partial T_i(r,t)}{\partial t}=a\left(\frac{\partial^2 T_i(r,t)}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial T_i(r,t)}{\partial r}\right)- -\tau_r\cdot\frac{\partial^2 T_i(r,t)}{\partial t^2},\ r>R,\ t>0, \tag{27}$$

$$T_i(r,t)\big|_{t=0}=T_0,\ r\ge R,\left.\frac{\partial T_i(r,t)}{\partial t}\right|_{t=0}=0,\quad r\ge R,\left|T_i(r,t)\right|<\infty,\ r\ge R,\ t\ge 0, \tag{28}$$

$$T_i(r,t)\big|_{r=R}=T_c,\ t>0,\quad \frac{1}{\tau_r}\int_0^t \left.\frac{\partial T_2(r,\tau)}{\partial r}\right|_{r=R}\exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau=-\frac{1}{\lambda_T}q_0,\ t>0, \tag{29}$$

$$\frac{1}{\tau_r}\int_0^t \left.\frac{\partial T_3(r,\tau)}{\partial r}\right|_{r=R}\exp\left(-\frac{t-\tau}{\tau_r}\right)d\tau= =h\left[T_3(r,t)\big|_{r=R}-T_c\right]t>0. \tag{30}$$

Such cases of intense heating of the surface area (real body) are of considerable practical interest, for example, in the following cases: surface dielectric heating; calculation of thermal stresses in the cylinder walls of steam machines and internal combustion engines; in the theory of automatic temperature control systems; in the studying the sound frequency region of metals at high or very low surface temperatures; numerous cases of abrupt change in the surface temperature of space and aviation objects; in the mechanical engineering industry when working at various experimental facilities to determine the temperature state of samples; etc. [6]. Of undoubted practical importance for the theory of thermal shock is representing all three cases simultaneously in three coordinate systems within the framework of the generalized model.

The following dimensionless variables may be introduced to avoid unnecessary inconvenience.

In region $z > R$, $t > 0$:

$$\left.\begin{aligned}
&\xi=\frac{v_{\text{p}}z}{a},\ \tau=\frac{v_{\text{p}}^2 t}{a},\ \xi_0=\frac{v_{\text{p}}R}{a},\ \beta=\frac{v_{\text{p}}}{v_{\text{T}}},\ Bi^*=\frac{ha}{v_{\text{p}}},\ S_{\text{T}}=\frac{2G\alpha_{\text{T}}(1+\nu)}{1-2\nu},\\
&W_i(\xi,\tau)=\frac{T_i(z,t)-T_0}{T_{\text{c}}-T_0},\ i=1,3;\ W_2(\xi,\tau)=\frac{T_2(z,t)-T_0}{(q_0/\lambda_{\text{T}})(a/v_{\text{p}})},\ i=2,\\
&\sigma_{\xi\xi}(\xi,\tau)=\frac{\sigma_{zz}(z,t)}{S_{\text{T}}(T_{\text{c}}-T_0)},\ i=1,3;\ \sigma_{\xi\xi}(\xi,\tau)=\frac{\sigma_{zz}(z,t)}{S_{\text{T}}(q_0/\lambda_{\text{T}})(a/v_{\text{p}})},\ i=2,\\
&U_i(\xi,\tau)=\frac{(1-\nu)U_z(z,t)}{(1+\nu)\alpha_{\text{T}}(T_{\text{c}}-T_0)(a/v_{\text{p}})},\ i=1,3,\\
&U_2(\xi,\tau)=\frac{(1-\nu)U_z(z,t)}{(1+\nu)\alpha_{\text{T}}(q_0/\lambda_{\text{T}})(a/v_{\text{p}})^2},\ i=2.
\end{aligned}\right\} \quad (31)$$

In region $\rho > R$, $t > 0$:

$$\left.\begin{aligned}
&\xi=\frac{v_{\text{p}}\rho}{a},\ \tau=\frac{v_{\text{p}}^2 t}{a},\ \xi_0=\frac{v_{\text{p}}R}{a},\ \beta=\frac{v_{\text{p}}}{v_{\text{T}}},\ Bi^*=\frac{ha}{v_{\text{p}}},\ S_{\text{T}}=\frac{2G\alpha_{\text{T}}(1+\nu)}{1-2\nu},\\
&W_i(\xi,\tau)=\frac{T_i(\rho,t)-T_0}{T_{\text{c}}-T_0},\ i=1,3;\ W_2(\xi,\tau)=\frac{T_2(\rho,t)-T_0}{(q_0/\lambda_{\text{T}})(a/v_{\text{p}})},\ i=2,\\
&\sigma_{\xi\xi}(\xi,\tau)=\frac{\sigma_{\rho\rho}(\rho,t)}{S_{\text{T}}(T_{\text{c}}-T_0)},\ i=1,3;\ \sigma_{\xi\xi}(\xi,\tau)=\frac{\sigma_{\rho\rho}(\rho,t)}{S_{\text{T}}(q_0/\lambda_{\text{T}})(a/v_{\text{p}})},\ i=2,\\
&U_i(\xi,\tau)=\frac{(1-\nu)U_\rho(\rho,t)}{(1+\nu)\alpha_{\text{T}}(T_{\text{c}}-T_0)(a/v_{\text{p}})},\ i=1,3,\\
&U_2(\xi,\tau)=\frac{(1-\nu)U_\rho(\rho,t)}{(1+\nu)\alpha_{\text{T}}(q_0/\lambda_{\text{T}})(a/v_{\text{p}})^2},\ i=2.
\end{aligned}\right\} \quad (32)$$

In region $r > R$, $t > 0$:

$$\left.\begin{aligned}
&\xi=\frac{v_{\text{p}}r}{a},\ \tau=\frac{v_{\text{p}}^2 t}{a},\ \xi_0=\frac{v_{\text{p}}R}{a},\ \beta=\frac{v_{\text{p}}}{v_{\text{T}}},\ Bi^*=\frac{ha}{v_{\text{p}}},\ S_{\text{T}}=\frac{2G\alpha_{\text{T}}(1+\nu)}{1-2\nu},\\
&W_i(\xi,\tau)=\frac{T_i(r,t)-T_0}{T_{\text{c}}-T_0},\ i=1,3;\ W_2(\xi,\tau)=\frac{T_2(r,t)-T_0}{(q_0/\lambda_{\text{T}})(a/v_{\text{p}})},\ i=2,\\
&\sigma_{\xi\xi}(\xi,\tau)=\frac{\sigma_{rr}(r,t)}{S_{\text{T}}(T_{\text{c}}-T_0)},\ i=1,3;\ \sigma_{\xi\xi}(\xi,\tau)=\frac{\sigma_{rr}(r,t)}{S_{\text{T}}(q_0/\lambda_{\text{T}})(a/v_{\text{p}})},\ i=2,\\
&U_i(\xi,\tau)=\frac{(1-\nu)U_r(r,t)}{(1+\nu)\alpha_{\text{T}}(T_{\text{c}}-T_0)(a/v_{\text{p}})},\ i=1,3,\\
&U_2(\xi,\tau)=\frac{(1-\nu)U_r(r,t)}{(1+\nu)\alpha_{\text{T}}(q_0/\lambda_{\text{T}})(a/v_{\text{p}})^2},\ i=2.
\end{aligned}\right\} \quad (33)$$

GENERALIZED MATHEMATICAL MODEL OF THE PROBLEM AND ITS ANALYTICAL SOLUTION

Assuming the boundaries of the region free of stresses, the generalized model of the problem may be written as follows:

$$\frac{\partial^2 U_i(\xi,\tau)}{\partial \xi^2}+\frac{2m+1}{\xi}\left[\frac{\partial U_i(\xi,\tau)}{\partial \xi}-\frac{U_i(\xi,\tau)}{\xi}\right]-$$
$$-\frac{\partial^2 U_i(\xi,\tau)}{\partial \tau^2}=\frac{\partial W_i(\xi,\tau)}{\partial \xi},\ \xi>\xi_0,\ \tau>0. \tag{34}$$

$$U_i(\xi,\tau)\big|_{\tau=0}=\left.\frac{\partial U_i(\xi,\tau)}{\partial \tau}\right|_{\tau=0}=0,$$
$$\xi\ge\xi_0,\ \left|U_i(\xi,\tau)\right|<\infty,\ \xi\ge\xi_0,\tau\ge 0. \tag{35}$$

$$\left[\frac{\partial U_i(\xi,\tau)}{\partial \xi}+\frac{(2m+1)v}{1-v}\cdot\frac{1}{\xi}U_i(\xi,\tau)\right]_{\xi=\xi_0}=$$
$$=W_i(\xi,\tau)\big|_{\xi=\xi_0},\ \tau>0. \tag{36}$$

$$\frac{\partial W_i(\xi,\tau)}{\partial \tau}=\frac{\partial^2 W_i(\xi,\tau)}{\partial \xi^2}+\xi\frac{\partial W_i(\xi,\tau)}{\partial \xi}-$$
$$-\beta^2\frac{\partial^2 W_i(\xi,\tau)}{\partial \tau^2}=0,\ \xi>\xi_0,\ \tau>0. \tag{37}$$

$$W_i(\xi,\tau)\big|_{\tau=0}=0,\ \xi\ge\xi_0,\ \left.\frac{\partial W_i(\xi,\tau)}{\partial \tau}\right|_{\tau=0}=0;$$
$$\left|W_i(\xi,\tau)\right|<\infty,\ \xi\ge\xi_o,\ \tau\ge 0. \tag{38}$$

$$W_1(\xi,\tau)\big|_{\xi=\xi_0}=1,\ \tau>0. \tag{39}$$

$$\frac{1}{\beta^2}\int_0^\tau \left.\frac{\partial W_2(\xi,\tau')}{\partial \xi}\right|_{\xi=\xi_0}\exp\left(-\frac{\tau-\tau'}{\beta^2}\right)d\tau'=-1,\ \tau>0. \tag{40}$$

$$\frac{1}{\beta^2}\int_0^\tau \left.\frac{\partial W_3(\xi,\tau')}{\partial \xi}\right|_{\xi=\xi_0}\exp\left(-\frac{\tau-\tau'}{\beta^2}\right)d\tau'=$$
$$=Bi^*\left[W_3(\xi,\tau)\big|_{\xi=\xi_0}=-1\right],\ \tau>0. \tag{41}$$

Here, $i=1$ at $m=-0.5$, $i=2$ at $m=0.5$, and $i=3$ at $m=0$. It should be noted that the desired stress $\sigma_{\xi\xi}(\xi,\tau)$ in all three cases is related to displacement $U_i(\xi,\tau)$ by the following relation:

$$\sigma_{\xi\xi}(\xi,\tau)=\frac{\partial U_i(\xi,\tau)}{\partial \xi}+\frac{(2m+1)v}{(1-v)}\times$$
$$\times\frac{1}{\xi}\cdot U_i(\xi,\tau)-W_i(\xi,\tau). \tag{42}$$

The solution of the generalized problem (34)–(42) in order to study, for example, the impact of the region geometry on the kinetics of corresponding thermoelastic stresses is an open problem of thermal shock theory. Its solution is given below.

According to Laplace, in the image space

$$\left.\begin{aligned}\overline{W}_i(\xi,p)&=\int_0^\infty W_i(\xi,\tau)\exp(-p\tau)d\tau,\\ \overline{U}_i(\xi,p)&=\int_0^\infty U_i(\xi,\tau)\exp(-p\tau)d\tau,\\ \overline{\sigma}_{\xi\xi}(\xi,p)&=\int_0^\infty \sigma_{\xi\xi}(\xi,\tau)\exp(-p\tau)d\tau,\end{aligned}\right\} \tag{43}$$

the solution of the thermal problem (37)–(41) may be written in the following form:

$$\overline{W}_i(\xi,p)=\overline{f}_i(\xi_0,p)(\xi_0/\xi)^m K_m\left[\xi\overline{\mu}(p)\right],$$
$$\left(\overline{\mu}(p)=\sqrt{\beta^2p^2+p}\right),$$

where

$$\overline{f}_i(\xi_0,p)=$$
$$=\begin{cases}\dfrac{1}{pK_m[\xi_0\overline{\mu}(p)]}, & i=1,\\[2mm] \dfrac{\overline{\mu}(p)}{p^2K_{m+1}[\xi_0\overline{\mu}(p)]}, & i=2,\\[2mm] \dfrac{Bi^*\overline{\mu}(p)}{p\{pK_{m+1}[\xi_0\overline{\mu}(p)]+Bi^*\overline{\mu}(p)K_m[\xi_0\overline{\mu}(p)]\}}, & i=3.\end{cases}$$

Here, $K_m(z)$ is the modified Bessel function.

We shall transform relations (34)–(36) into the image space (43):

$$\frac{d^2\overline{U}_i(\xi,p)}{d\xi^2}+\frac{2m+1}{\xi}\cdot\frac{d\overline{U}_i(\xi,p)}{d\xi}-$$
$$-\left(p^2+\frac{2m+1}{\xi^2}\right)\overline{U}_i(\xi,p)=\frac{d\overline{W}_i(\xi,p)}{d\xi},\ \xi>\xi_0. \tag{44}$$

$$\left[\frac{d\overline{U}_i(\xi,p)}{d\xi}+\frac{(2m+1)v}{(1-v)}\cdot\frac{1}{\xi}\cdot\overline{U}_i(\xi,p)\right]_{\xi=\xi_0}=$$
$$=\overline{W}_i(\xi,p)\big|_{\xi=\xi_0}. \tag{45}$$

$$\left|\overline{U}_i(\xi,p)\right|<\infty,\ \xi\ge\xi_0. \tag{46}$$

The general solution of equation (44) consists of the general solution of homogeneous equation and the partial solution of inhomogeneous equation:

$$\overline{U}_i(\xi,p)=\frac{1}{\xi^m}\left[C_1K_{m+1}(\xi p)+C_2I_{m+1}(\xi p)\right]+ \\ +\overline{U}_{i\text{part}}(\xi,p). \tag{47}$$

Finding a partial solution of Eq. (44) requires a separate consideration. We have:

$$\frac{d}{d\xi}(\Delta\overline{U}_i)=\Delta\left(\frac{d\overline{U}_i}{d\xi}\right)-\frac{2m+1}{\xi^2}\cdot\frac{d\overline{U}_i}{d\xi}. \tag{48}$$

Here,

$$\Delta\overline{U}_i=\frac{d^2\overline{U}_i}{d\xi^2}+\frac{2m+1}{\xi}\cdot\frac{d\overline{U}_i}{d\xi}, \\ \Delta\overline{W}_i=(\beta^2p^2+p)\overline{W}_i. \tag{49}$$

The partial solution in (47) is found in the form $\overline{U}_{i\text{part}}(\xi,p)=A\dfrac{d\overline{W}_i}{d\xi}$, where constant A is to be found. Substituting this expression into (44) and using (48)–(49), $A=\dfrac{1}{p^2(\beta^2-1)+p}$ is found. Thus, the general solution of Eq. (44) with allowance for boundary condition (46) may be written in the following form:

$$\overline{U}_i(\xi,p)=C\left[\frac{1}{\xi^m}K_{m+1}(\xi p)\right]- \\ -\overline{\varphi}_i(\xi_0,p)(\xi_0/\xi)^mK_{m+1}\left[\xi\overline{\mu}(p)\right], \tag{50}$$

where

$$\overline{\varphi}_i(\xi_0,p)= \\ =\begin{cases}\dfrac{\overline{\mu}(p)}{p^2[p(\beta^2-1)+1]K_m[\xi_0\overline{\mu}(p)]}, & i=1,\\ \dfrac{\overline{\mu}^2(p)}{p^3[p(\beta^2-1)+1]K_{m+1}[\xi_0\overline{\mu}(p)]}, & i=2,\\ \dfrac{Bi^*\overline{\mu}^2(p)}{p^2[p(\beta^2-1)+1][pK_{m+1}(\xi_0\overline{\mu}(p))+Bi^*\overline{\mu}(p)K_m(\xi_0\overline{\mu}(p))]}, & i=3.\end{cases}$$

The following relation is used here:

$$\frac{d}{d\xi}\left[\xi^{-m}K_m(\xi\overline{\mu}(p))\right]=-\overline{\mu}(p)\xi^{-m}K_{m+1}\left[\xi\overline{\mu}(p)\right].$$

Using the properties of Bessel functions, the following important relation being of fundamental importance for finding an analytical solution of the generalized problem (34)–(42) is found after transformations:

$$\frac{d}{d\xi}\left[\frac{1}{\xi^m}K_{m+1}(\xi p)\right]+\frac{(2m+1)\nu}{(1-\nu)}\times \\ \times\frac{1}{\xi^{m+1}}\cdot K_{m+1}(\xi p)=-\frac{1}{\xi^{m+1}}\overline{\Theta}(\xi p), \tag{51}$$

where

$$\overline{\Theta}(\xi p)=\xi pK_m(\xi p)+(2m+1)\frac{(1-2\nu)}{(1-\nu)}K_{m+1}(\xi p). \tag{52}$$

Relations (59)–(60) and the boundary condition (50) allow determining the constant in (55) along with the desired displacement. We find:

$$\overline{U}_i(\xi,p)=\frac{(\xi_0/\xi)^m\overline{\Theta}\left[\xi_0\mu(p)\right]K_{m+1}(\xi p)\overline{\varphi}_i(\xi_0p)}{\overline{\Theta}(\xi_0p)}- \\ -\frac{(\xi_0/\xi)^mK_m\left[\xi_0\overline{\mu}(p)\right]K_{m+1}(\xi p)\overline{f}_i(\xi_0,p)}{\xi_0^{-1}\overline{\Theta}(\xi_0p)}- \\ -(\xi_0/\xi)^m\overline{\varphi}_i(\xi_0,p)K_{m+1}\left[\xi\overline{\mu}(p)\right].$$

The desired stress is found, as follows:

$$\frac{\overline{\sigma}_{\xi\xi}(\xi,p)}{(\xi_0/\xi)^m}=\frac{(\xi_0/\xi)\overline{\Theta}(\xi p)K_m\left[\xi_0\overline{\mu}(p)\right]\overline{f}_i(\xi_0,p)}{\overline{\Theta}(\xi_0p)}+ \\ +\frac{\overline{\varphi}_i(\xi_0,p)\overline{\Theta}\left[\xi\overline{\mu}(p)\right]}{\xi}-\frac{\overline{\varphi}_i(\xi_0,p)\overline{\Theta}\left[\xi_0\overline{\mu}(p)\right]\overline{\Theta}(\xi p)}{\xi\overline{\Theta}(\xi_0p)}- \\ -\overline{f}_i(\xi_0,p)K_m\left[\xi\overline{\mu}(p)\right].$$

The following operational relations for stresses for all three types of heating may be written:

- temperature heating ($i = 1$)

$$\frac{\overline{\sigma}_{\xi\xi}(\xi,p)}{(\xi_0/\xi)^m}=\frac{(\xi_0/\xi)\overline{\Theta}(\xi p)}{p\overline{\Theta}(\xi_0p)}+ \\ +\frac{\overline{\mu}(p)\overline{\Theta}\left[\xi\overline{\mu}(p)\right]}{\xi p^2[p(\beta^2-1)+1]K_m[\xi_0\overline{\mu}(p)]}- \\ -\frac{K_m\left[\xi\overline{\mu}(p)\right]}{pK_m[\xi_0\overline{\mu}(p)]}- \\ -\frac{\overline{\mu}(p)\overline{\Theta}\left[\xi_0\overline{\mu}(p)\right]\overline{\Theta}(\xi p)}{\xi p^2[p(\beta^2-1)+1]K_m[\xi_0\overline{\mu}(p)]\overline{\Theta}(\xi_0p)}; \tag{53}$$

- thermal heating ($i = 2$)

$$\frac{\overline{\sigma}_{\xi\xi}(\xi,p)}{(\xi_0/\xi)^m}=\frac{(\xi_0/\xi)\overline{\mu}(p)\overline{\Theta}(\xi p)K_m[\xi_0\overline{\mu}(p)]}{p^2\overline{\Theta}(\xi_0p)K_{m+1}[\xi_0\overline{\mu}(p)]}+ \\ +\frac{\overline{\mu}^2(p)\overline{\Theta}(\xi\overline{\mu}(p))}{\xi p^3[p(\beta^2-1)+1]K_{m+1}[\xi_0\overline{\mu}(p)]}- \\ -\frac{\overline{\mu}(p)K_m[\xi_0\overline{\mu}(p)]}{p^2K_{m+1}[\xi_0\overline{\mu}(p)]}- \\ -\frac{\overline{\mu}^2(p)\overline{\Theta}(\xi p)\overline{\Theta}[\xi_0\overline{\mu}(p)]}{\xi p^3[p(\beta^2-1)+1]\overline{\Theta}(\xi_0p)K_{m+1}[\xi_0\overline{\mu}(p)]}; \tag{54}$$

- heating by medium ($i = 3$)

$$\frac{\bar{\sigma}_{\xi\xi}(\xi,p)}{(\xi_0/\xi)^m}=\frac{(\xi_0/\xi)\bar{\Theta}(\xi p)}{p\bar{\Theta}(\xi_0 p)}\times$$
$$\times\frac{K_m[\xi_0\bar{\mu}(p)]Bi^*\bar{\mu}(p)}{[pK_{m+1}(\xi_0\bar{\mu}(p))+Bi^*\bar{\mu}(p)K_m(\xi_0\bar{\mu}(p))]\bar{\mu}}+$$
$$+\frac{Bi^*\bar{\mu}^2(p)}{\xi p^2[p(\beta^2-1)+1]}\times$$
$$\times\frac{\bar{\Theta}[\xi_0\bar{\mu}(p)]}{[pK_{m+1}(\xi_0\bar{\mu}(p))+Bi^*\bar{\mu}(p)K_m(\xi_0\bar{\mu}(p))]}- \quad (55)$$
$$-\frac{Bi^*\bar{\mu}^2(p)}{\xi p^2[p(\beta^2-1)+1]}\times$$
$$\times\frac{\bar{\Theta}[\xi_0\bar{\mu}(p)]\bar{\Theta}(\xi p)}{[pK_{m+1}(\xi_0\bar{\mu}(p))+Bi^*\bar{\mu}(p)K_m(\xi_0\bar{\mu}(p))]\bar{\Theta}(\xi_0 p)}-$$
$$-\frac{Bi^*\bar{\mu}(p)K_m[\xi_0\bar{\mu}(p)]}{p[pK_{m+1}(\xi_0\bar{\mu}(p))+Bi^*\bar{\mu}(p)K_m(\xi\bar{\mu}(p))]}.$$

The inverse transformation in (53)–(55) turns out to be rather cumbersome resulting in the complex and hard to see expressions. But if inertial effects of a microsecond duration [17] during which stresses $\sigma_{\xi\xi}(\xi, \tau)$ just reach maximum values are considered in formulating problem (34)–(42), then short times τ corresponding to large p in images, that is immediately after the thermal shock, may be considered. For this, the asymptotic representation of cylindrical functions for large p is used in (53)–(55), as follows:

$$K_\nu(p)=\frac{\sqrt{\pi/2}}{\sqrt{p}}\exp(-p),\ \nu\ge 0,$$

where K_ν is the modified Bessel function.

The following notations are introduced then:

$$A_m(\xi)=\frac{(2m+1)(1-2\nu)}{(1-\nu)\xi},\ \gamma=\frac{1}{1-\beta^2}.$$

We shall also introduce a function for this class of problems that is the Kartashov function

$$\bar{\chi}_m(\xi,p)=\frac{p+A_m(\xi)}{(p-\gamma)[p+A_m(\xi_0)]},$$

which is used in finding the originals of found images. The original of this function may be written in the following form:

$$\chi_m(\xi,\tau)=\frac{\gamma+A_m(\xi)}{\gamma+A_m(\xi_0)}\times$$
$$\times\left[\frac{A_m(\xi_0)-A_m(\xi)}{\gamma+A_m(\xi)}\exp(-A_m(\xi_0)\tau)+\exp(\gamma\tau)\right].$$

After transformations, we find:

$$\frac{\bar{\sigma}_{\xi\xi}(\xi,p)}{(\xi_0/\xi)^{m+1/2}}=-\bar{\Psi}_1(\xi,p)\exp\left[-(\xi-\xi_0)\bar{\mu}(p)\right]+$$
$$+\bar{\Psi}_2(\xi,p)\exp\left[-(\xi-\xi_0)p\right], \quad (56)$$

where at temperature heating ($i = 1$):

$$\bar{\Psi}_1(\xi,p)=\left\{\frac{\gamma\bar{\mu}(p)[\bar{\mu}(p)+A_m(\xi)]}{p^2(p-\gamma)}+\frac{1}{p}\right\}, \quad (57)$$

$$\bar{\Psi}_2(\xi,p)=\left\{\frac{p-\gamma}{p}\bar{\chi}_m(\xi,p)+\right.$$
$$\left.+\frac{\gamma\bar{\mu}(p)\left[\bar{\mu}(p)+A_m(\xi_0)\right]\bar{\chi}_m(\xi,p)}{p^2}\right\}; \quad (58)$$

at thermal heating ($i = 2$):

$$\bar{\Psi}_1(\xi,p)=\left\{\frac{\gamma\bar{\mu}^2(p)[\bar{\mu}(p)+A_m(\xi)]}{p^3(p-\gamma)}+\frac{\bar{\mu}^2(p)}{p^2}\right\}, \quad (59)$$

$$\bar{\Psi}_2(\xi,p)=\left\{\frac{\bar{\mu}(p)(p-\gamma)\bar{\chi}_m(\xi,p)}{p^2}+\right.$$
$$\left.+\frac{\gamma\bar{\mu}^2(p)[\bar{\mu}(p)+A_m(\xi_0)]\bar{\chi}_m(\xi,p)}{p^3}\right\}; \quad (60)$$

at heating by medium ($i = 3$):

$$\bar{\Psi}_1(\xi,p)=$$
$$=\left\{\frac{\gamma Bi^*\bar{\mu}^2(p)[\bar{\mu}(p)+A_m(\xi)]}{p^2(p-\gamma)[p+Bi^*\bar{\mu}(p)]}+\right. \quad (61)$$
$$\left.+\frac{Bi^*\bar{\mu}(p)}{p[p+Bi^*\bar{\mu}(p)]}\right\},$$

$$\bar{\Psi}_2(\xi,p)=\left\{\frac{Bi^*\bar{\mu}(p)(p-\gamma)\bar{\chi}_m(\xi,p)}{p[p+Bi^*\bar{\mu}(p)]}+\right.$$
$$\left.+\frac{\gamma Bi^*\bar{\mu}^2(p)[\bar{\mu}(p)+A_m(\xi_0)]\bar{\chi}_m(\xi,p)}{p^2[p+Bi^*\bar{\mu}(p)]}\right\}. \quad (62)$$

When finding the originals in (56)–(62), attention should be paid to the value of parameter $\beta = v_p/v_T$. Thus, $\beta = 0.4$ for organic glass and $\beta = 0.7$ for quartz and silicon, that is $\beta < 1$; $\beta = 3.4$ for steel and $\beta = 1.8$ for crystals and aluminum, that is $\beta > 1$. The value of parameter β plays the determining role in writing stress intervals $\sigma_{\xi\xi}(\xi, \tau)$. The following is found for stresses:

at $\beta < 1$:

$$\frac{\sigma_{\xi\xi}(\xi,\tau)}{(\xi_0/\xi)^{m+1/2}} = \begin{cases} 0, & \tau < (\xi-\xi_0)\beta, \\ -\sigma_{\xi\xi}^{(1)}(\xi,\tau), & (\xi-\xi_0)\beta < \tau < (\xi-\xi_0), \\ -\sigma_{\xi\xi}^{(1)}(\xi,\tau)+\sigma_{\xi\xi}^{(2)}(\xi,\tau), & \tau > (\xi-\xi_0); \end{cases} \quad (63)$$

at $\beta > 1$:

$$\frac{\sigma_{\xi\xi}(\xi,\tau)}{(\xi_0/\xi)^{m+1/2}} = \begin{cases} 0, & \tau < (\xi-\xi_0), \\ \sigma_{\xi\xi}^{(2)}(\xi,\tau), & (\xi-\xi_0) < \tau < (\xi-\xi_0)\beta, \\ -\sigma_{\xi\xi}^{(1)}(\xi,\tau)+\sigma_{\xi\xi}^{(2)}(\xi,\tau), & \tau > \beta(\xi-\xi_0). \end{cases} \quad (64)$$

Here, $\sigma_{\xi\xi}^{(1)}(\xi,\tau)$ is the original mage

$$\overline{\Psi}_1(\xi,p)\exp\left[-(\xi-\xi_0)\beta\sqrt{p^2+p/\beta^2}\right],\ \sigma_{\xi\xi}^{(2)}(\xi,\tau)=\Psi_2[\xi,\tau-(\xi-\xi_0)].$$

Turning to the originals, the following analytical relations for stresses in all three cases of heating may be written:

- temperature heating ($i = 1$)

$$\begin{aligned} \sigma_{\xi\xi}^{(1)}(\xi,\tau) &= \gamma\Big\langle \exp\{\gamma[\tau-(\xi-\xi_0)\beta]-(\xi-\xi_0)/2\beta\} + \\ &+\frac{\xi-\xi_0}{2\beta}\int_{(\xi-\xi_0)\beta}^{\tau}\exp[\gamma(\tau-\tau')-(1/2\beta^2)\tau']\frac{I_1\left((1/2\beta^2)\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}\right)}{\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}}d\tau'\Big\rangle\times\eta[\tau-(\xi-\xi_0)\beta]+ \\ &+\frac{\gamma A_m(\xi)}{\beta}\left\{\int_{(\xi-\xi_0)\beta}^{\tau}\exp\left[\gamma(\tau-\tau')-(1/2\beta^2)\tau'\right]I_0\left((1/2\beta^2)\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}\right)d\tau'\right\}\times\eta[\tau-(\xi-\xi_0)\beta]- \\ &-\frac{A_m(\xi)}{\beta}\left\{\int_{(\xi-\xi_0)\beta}^{\tau}\exp[-(1/2\beta^2)\tau']I_0\left((1/2\beta^2)\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}\right)d\tau'\right\}\times\eta[\tau-(\xi-\xi_0)\beta]. \end{aligned} \quad (65)$$

$$\sigma_{\xi\xi}^{(2)}(\xi,\tau)=\Psi_2[\xi,\tau-(\xi-\xi_0)], \quad (66)$$

where

$$\Psi_2(\xi,\tau)=\gamma\chi_m(\xi,\tau)+A_m(\xi_0)\left[2\beta\int_0^{\tau}\chi_m(\xi,\tau')d\tau'\right]-\beta\int_0^{\tau}\exp[-(1/2\beta^2)(\tau-\tau')]\chi_m(\xi,\tau')d\tau'; \quad (67)$$

- thermal heating ($i = 2$)

$$\begin{aligned} \sigma_{\xi\xi}^{(1)}(\xi,\tau) &= \gamma\beta\exp[-(1/2\beta^2)\tau]I_0\left((1/2\beta^2)\sqrt{\tau^2-(\xi-\xi_0)^2\beta^2}\right)\eta[\tau-(\xi-\xi_0)\beta]+ \\ &+\frac{\gamma^2}{\beta}\left\{\int_{(\xi-\xi_0)\beta}^{\tau}\exp[\gamma(\tau-\tau')-(1/2\beta^2)\tau']I_0\left((1/2\beta^2)\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}\right)d\tau'\right\}\times\eta[\tau-(\xi-\xi_0)\beta]+ \\ &+A_m(\xi)\Big\langle\Big\{\exp\gamma[\tau-(\xi-\xi_0)\beta]-1-[\tau-(\xi-\xi_0)\beta]\Big\}\exp(-1/2\beta^2)\tau+ \\ &+\frac{\xi-\xi_0}{2\beta}\int_{(\xi-\xi_0)\beta}^{\tau}[\exp\gamma(\tau-\tau')-1-(\tau-\tau')]\exp[-(1/2\beta^2)]\times \\ &\times\frac{I_1\left((1/2\beta^2)\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}\right)}{\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}}d\tau'\Big\rangle\eta[\tau-(\xi-\xi_0)\beta]. \end{aligned} \quad (68)$$

$$\Psi_2(\xi,\tau)=\gamma\beta\chi_m(\xi,\tau)+\frac{\gamma[1+2\beta^3A_m(\xi_0)]}{2\beta}\int_0^\tau\chi_m(\xi,\tau')d\tau'+\gamma A_m(\xi_0)\int_0^\tau(\tau-\tau')\chi_m(\xi,\tau')d\tau'; \quad (69)$$

- heating by medium ($i=3$). For this case, decomposition of the image $\bar{\mu}(p)=\beta p[1+1/(\beta^2p)]^{1/2}$ into the binominal series is used in finding the original. We shall also introduce the following number of notations:

$$\gamma=1+\gamma\beta^2;\ \gamma_1=\frac{1}{1+Bi^*\beta};\ \gamma_2=\frac{Bi^*}{2\beta(1+Bi^*\beta)};$$

$$A_1=\gamma-\frac{\gamma_1[\gamma^2+(\gamma-1)Bi^*A_m(\xi)]}{\gamma+\gamma_2};$$

$$A_2=\frac{\gamma_1[\gamma^2+(\gamma-1)Bi^*A_m(\xi)]}{\gamma+\gamma_2}-\gamma\gamma_1+2\beta A_m(\xi);\ A_3=-2\beta A_m(\xi);$$

$$A_4=[\gamma\gamma_1(1+\gamma_2)-\gamma_1\gamma_2-\gamma(1+\gamma_1)\beta A_m(\xi_0)].$$

The desired originals are found, as follows:

$$\begin{aligned}
\Psi_1(\xi,\tau)&=A_1\left\{\exp[\gamma(\tau-(\xi-\xi_0)\beta)-(\xi-\xi_0)/2\beta]\right\}+\\
&+\frac{A_1(\xi-\xi_0)}{2\beta}\int_{(\xi-\xi_0)\beta}^{\tau}\exp\left[\gamma(\tau-\tau')-(1/2\beta^2)\tau'\right]\frac{I_1\left((1/2\beta^2)\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}\right)}{\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}}d\tau'\times\eta[\tau-(\xi-\xi_0)\beta]+\\
&+A_2\left\langle\exp[-\gamma_2(\tau-(\xi-\xi_0)\beta))-(\xi-\xi_0/2\beta]+\frac{\xi-\xi_0}{2\beta}\int_{(\xi-\xi_0)\beta}^{\tau}\exp[-\gamma_2(\tau-\tau')-(1/2\beta^2)\tau']\times\right.\\
&\left.\times\frac{I_1\left((1/2\beta^2)\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}\right)}{\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}}d\tau'\right\rangle\times\eta[\tau-(\xi-\xi_0)\beta]+\\
&+A_3\times\left\{\exp[-(1/2\beta)(\xi-\xi_0)]+\frac{\xi-\xi_0}{2\beta}\int_{(\xi-\xi_0)\beta}^{\tau}\exp[-(1/2\beta^2)\tau']\frac{I_1\left((1/2\beta^2)\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}\right)}{\sqrt{(\tau')^2-(\xi-\xi_0)^2\beta^2}}d\tau'\right\}\times\\
&\times\eta[\tau-(\xi-\xi_0)\beta];
\end{aligned} \quad (70)$$

$$\Psi_2(\xi,\tau)=\gamma(1-\gamma_1)\chi_m(\xi,\tau)+2\beta A_m(\xi_0)\int_0^\tau\chi_m(\xi,\tau')d\tau'+A_4\int_0^\tau\exp\left[-\gamma_2(\tau-\tau')\right]\chi_m(\xi,\tau')d\tau'. \quad (71)$$

Thus, the resulted relations (63)–(71) complete solving the practically important problem on the thermal reaction of the massive body with internal cavities. In addition to the developed approach, the highly effective method of estimating the value of thermal stresses practically not applied earlier in the theory of thermal shock may be specified.

As follows from the operational solution of the dynamic problem, the presence of term $\overline{\Psi}_2(\xi,p)\exp\left[-(\xi-\xi_0)p\right]$ in (56) shows the possibility of proposing the computational engineering formula for the upper-bound estimate of the temperature stress through the stress jump in (70) at the front of the thermoelastic wave. For this, the lag theorem [6] is used in the following form:

$$\begin{cases}\overline{\Psi}(\xi,p)\exp\left[-(\xi-\xi_0)p\right]\leftarrow 0, & \tau<\xi-\xi_0,\\ \Psi_2\left[\xi,\tau-(\xi-\xi_0)\right], & \tau>\xi-\xi_0,\end{cases}$$

wherefrom it is seen that at point ($\xi-\xi_0$), a jump of function $\Psi_2(\xi,\tau)$ occurs, whose value is calculated by the following equation:

$$|\Delta|=\lim_{\tau\to(\xi-\xi_0)+0}\Psi_2\left[\xi,\tau-(\xi-\xi_0)\right]=$$
$$=\lim_{\tau\to 0+}\Psi_2(\xi,\tau)=\lim p_{p\to\infty}\overline{\Psi}_2(\xi,p).$$

We shall first find the jump value for stresses $\sigma_{\xi\xi}(\xi,\tau)$ in coordinates (ξ,τ), then we pass to initial stresses in initial regions (11)–(30) using transfer equations (31)–(33). As a result, the following is found for temperature heating (58):

$$|\Delta| = \left|\sigma_{zz};\sigma_{\rho\rho};\sigma_{rr}\right|_{\max} = \begin{cases} \dfrac{2G\gamma(1+\nu)\alpha_T\left|T_c - T_0\right|}{(1-2\nu)}, & m=-1/2; \\ \dfrac{(R/\rho)2G\gamma(1+\nu)\alpha_T\left|T_c - T_0\right|}{(1-2\nu)}, & m=1/2; \\ \dfrac{\sqrt{R/r}\,2G\gamma(1+\nu)\alpha_T\left|T_c - T_0\right|}{(1-2\nu)}, & m=0. \end{cases} \quad (72)$$

The estimates (72) are valid both for intense heating ($T_c > T_0$) when compressive stresses occur in the fixed cross-section $\xi = \text{const} > \xi_0$, and for cooling ($T_c < T_0$) when more dangerous tensile stresses occur.

PHYSICAL ANALYSIS OF THE SOLUTION

Figure 1 shows the stress (63), (65)–(67) (temperature heating) vs. time curves in the fixed cross section $\xi = 1$ with $\xi_0 = 0.1$, $v = 0.3$, $\beta = 0.4$ for $m = -0.5$; 0; 0.5. The curves for stresses (temperature heating) under the same data but for $\beta = 0$, i.e., for the case of classical Fourier phenomenology based on parabolic type equations, are shown in Fig. 2.

It would be of interest to compare these two approaches, namely to describe the effect of hyperbolicity of the dynamic problem (i.e., the effect of local disequilibrium in the system) on the value of temperature stresses. We shall write out the solution of the "parabolic" problem of thermal shock under temperature heating for all three areas described above, considering that this part of the research is also of great practical interest and is also an open problem so far.

$$\frac{\sigma_{\xi\xi}(\xi,\tau)}{(\xi_0/\xi)^{m+1/2}} = \sigma_{\xi\xi}^{(1)}(\xi,\tau) + \begin{cases} 0, & \tau < \xi - \xi_0, \\ \sigma_{\xi\xi}^{(2)}(\xi,\tau), & \tau > \xi - \xi_0, \end{cases} \quad (73)$$

where

$$\sigma_{\xi\xi}^{(1)}(\xi,\tau) = -\Psi_1^*(\xi,\tau); \quad \sigma_{\xi\xi}^{(2)}(\xi,\tau) = \Psi_2^*[\xi,\tau-(\xi-\xi_0)];$$

$$\Psi_1^*(\xi,\tau) = \frac{1}{2}\left\{\exp\left[\tau-(\xi-\xi_0)\right]\Phi^*\left(\frac{\xi-\xi_0}{2\sqrt{\tau}}-\sqrt{\tau}\right) + \right.$$
$$\left. + \exp\left[\tau+(\xi-\xi_0)\right]\Phi^*\left(\frac{\xi-\xi_0}{2\sqrt{\tau}}+\sqrt{\tau}\right)\right\} +$$
$$+ A_m(\xi)\left\langle \frac{1}{2}\left[\exp(\tau-(\xi-\xi_0))\Phi^*\left(\frac{\xi-\xi_0}{2\sqrt{\tau}}-\sqrt{\tau}\right) - \right.\right. \quad (74)$$
$$\left. - \exp(\tau+(\xi-\xi_0))\Phi^*\left(\frac{\xi-\xi_0}{2\sqrt{\tau}}+\sqrt{\tau}\right)\right] -$$
$$\left. - \frac{2\sqrt{\tau}}{\sqrt{\pi}}\exp\left[-\frac{(\xi-\xi_0)^2}{4\tau}\right] + (\xi-\xi_0)\Phi^*\left(\frac{\xi-\xi_0}{2\sqrt{\tau}}\right)\right\rangle;$$

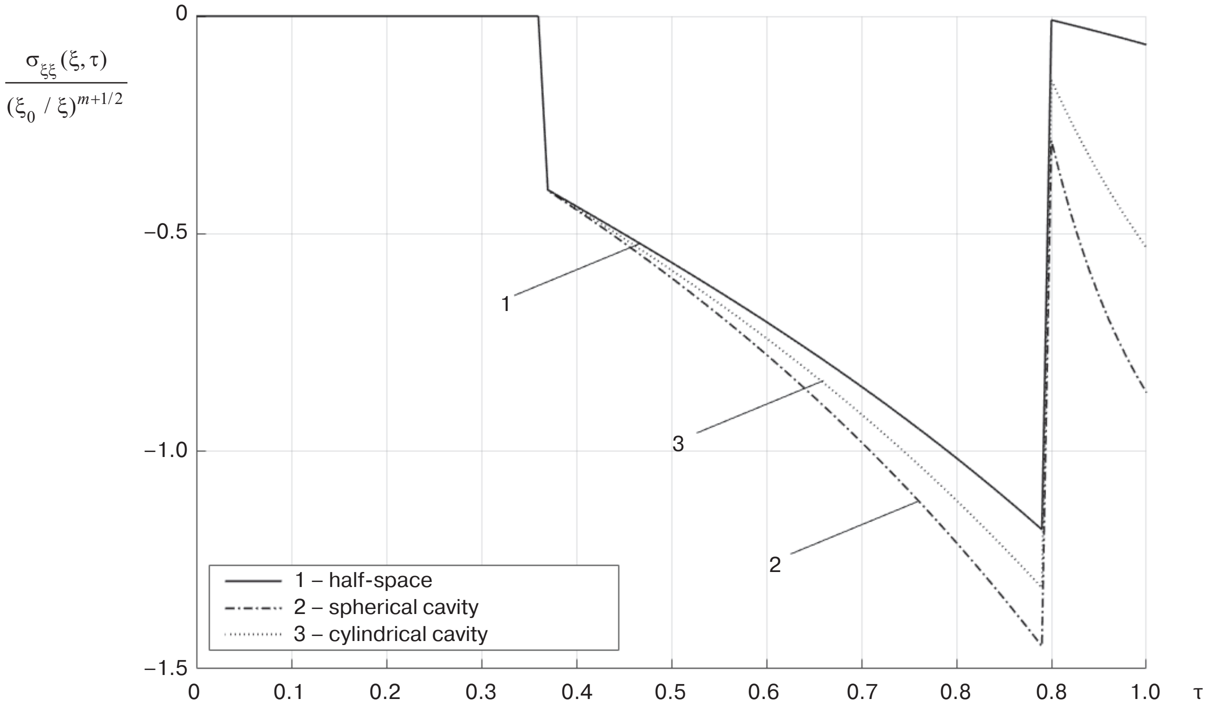

Fig. 1. Stress–time dependence in section $\xi = 1$ at $\xi_0 = 0.1$, $v = 0.3$, and $\beta = 0.4$. Calculated by equations (63), (65)–(67) for $m = -0.5$ (curve 1), $m = 0.5$ (curve 2), and $m = 0$ (curve 3)

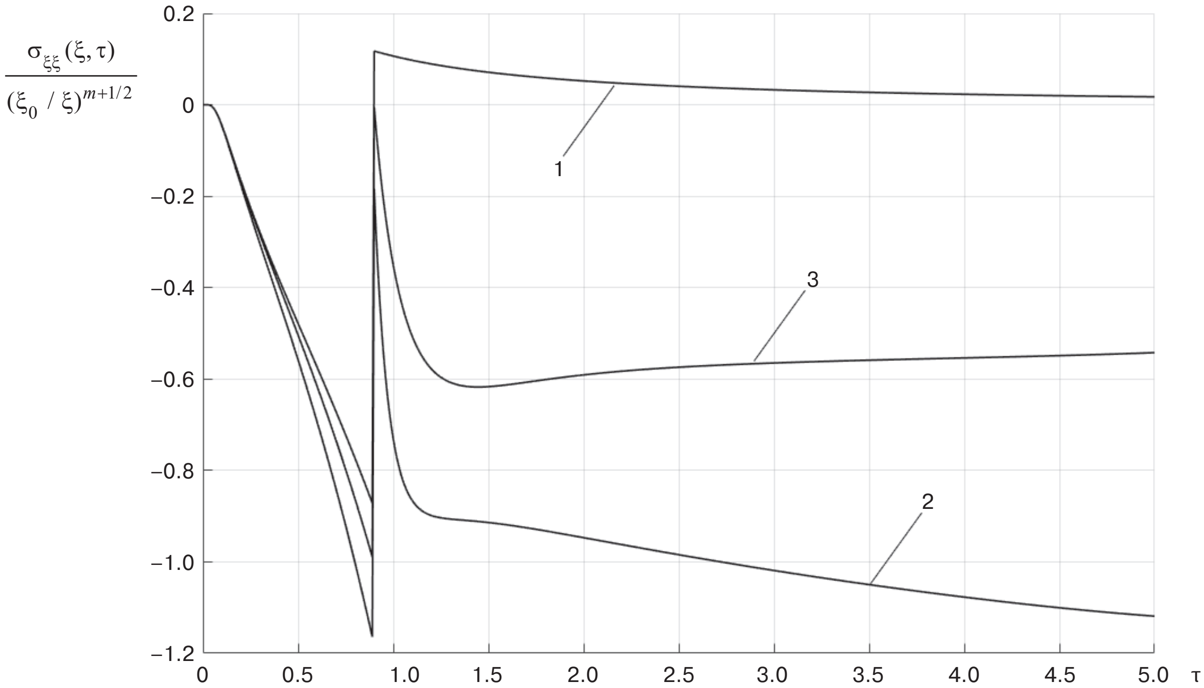

Fig. 2. Stress–time dependence in section ξ = 1 at ξ_0 = 0.1, *v* = 0.3, and (β = 0). Calculated by equations (73)–(75) for *m* = –0.5 (curve 1), *m* = 0.5 (curve 2), and *m* = 0 (curve 3)

$$\Psi_2^*(\xi,\tau) = \chi_m(\xi,\tau) + \frac{2A_m(\xi_0)}{\sqrt{\pi}}\int_0^\tau \sqrt{\tau-\tau'}\cdot\chi_m(\xi,\tau')\,d\tau'. \quad (75)$$

Here,

$$\Phi^*(z) = 1-\Phi(z),\ \Phi(z) = (2/\sqrt{\pi})\int_0^z \exp(-y^2)dy$$

is the Laplace function.

As it follows from the graphs, considering the finite speed of heat propagation results in a significant change in the pattern of dynamic temperature stresses compared to the data in Fig. 2. In contrast to (73), relation (63) shows the presence of two stress jumps, one occurring at the front of the heat wave and the other occurring at the front of the elastic wave moving at speeds v_T and v_p, respectively.

Let us consider the fixed cross-section inside region $\xi > \xi_0$. At $\beta < 1$, stresses in the cross-section are zero at the beginning. At the moment $\tau = \beta(\xi - \xi_0)$, the thermal stress wave arrives at this cross-section with its front moving at speed v_T; the compressive stress occurs jumping and increases further. At the moment $\xi - \xi_0$, the longitudinal elastic wave approaches the cross-section, thus causing a jump change in the stress and its further decrease.

The curves in Fig. 1 also show another interesting result: the most "vulnerable" (in terms of thermal reaction) is the region with inner spherical cavity. In all three cases ($m = -0.5$; 0; 0.5) the resulting stresses are compressive. Estimates (72) clearly show that the near-surface layers (near the boundary of the body (region)) are most susceptible to thermal shock. As for the thermal shock in the classical case (Fig. 2), the highest compression stresses are characteristic of the massive body bounded by flat surface ($m = -0.5$). In this case, after passing the expansion wave (due to function $\sigma_{\xi\xi}^{(2)}(\xi,\tau)$ in (72)), the stress passes into the region of positive (tensile) values and then rapidly decreases reaching quasi-static values.

CONCLUSIONS

The developed generalized model represents thermal shock in terms of dynamic thermoelasticity to describe locally nonequilibrium processes of heat transfer. An analytical solution of the (open) generalized problem of the thermal reaction of massive bodies with internal cavities simultaneously in Cartesian, cylindrical, and spherical coordinate systems under temperature heating, thermal heating, and heating the boundary of solid by medium is obtained. Numerical experiments showing the effect of local nonequilibrium on the kinetics of temperature stresses (compared to the classical Fourier phenomenology) have been carried out. The calculated engineering relations for the upper boundary of temperature stresses during intense temperature heating have important practical applications.

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Translated from Russian into English by Kirill V. Nazarov
Edited for English language and spelling by Thomas A. Beavitt