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RESEARCH ARTICLE

Spline approximation of multivalued functions in linear structures routing

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Abstract

Objectives. The theory and methods of spline approximation of plane curves given by a sequence of points are currently undergoing rapid development. Despite fundamental differences between used splines and those considered in the theory and its applications, results published earlier demonstrate the possibility of using spline approximation when designing routes of linear structures. The main difference here consists in the impossibility of assuming in advance the number of spline elements when designing the routes. Here, in contrast to widely use polynomial splines, the repeating element is the link "segment of a straight line + arc of a circle" or "segment of a straight line + arc of a clothoid + arc of a circle + arc of a clothoid." Previously, a two-stage scheme consisting of a determination of the number of elements of the desired spline and subsequent optimization of its parameters was proposed. Although an algorithm for solving the problem in relation to the design of a longitudinal profile has been implemented and published, this is not suitable for designing a route plan, since, unlike a profile, a route plan is generally a multivalued function. The present paper aims to generalize the algorithm for the case of spline approximation of multivalued functions making allowance for the design features of the routes of linear structures.

Methods. At the first stage, a novel mathematical model is developed to apply the dynamic programming method taking into account the constraints on the desired spline parameters. At the second stage, nonlinear programming is used. In this case, it is possible to analytically calculate the derivatives of the objective function with respect to the spline parameters in the absence of its analytical expression through these parameters.

Results. An algorithm developed for approximating multivalued functions given by a discrete series of points using a spline consisting of arcs of circles conjugated by line segments for solving the first stage of the problem is presented. An additional nonlinear programming algorithm was also used to optimize the parameters of the resulting spline as an initial approximation. However, in the present paper, the first stage is considered only, since the complex algorithm of the second stage and its justification require separate consideration.

Conclusions. The presented two-stage spline approximation scheme with an unknown number of spline elements is also suitable for approximating multivalued functions given by a sequence of points on a plane, in particular, for designing a route plan for linear structures.

Keywords: route, plan, longitudinal profile, spline, dynamic programming, objective function, constraints

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НАУЧНАЯ СТАТЬЯ

Сплайн-аппроксимация многозначных функций в проектировании трасс линейных сооружений

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Резюме

Цели. В настоящее время наблюдается бурное развитие теории и методов сплайн-аппроксимации плоских кривых, заданных последовательностью точек. Проведенные исследования, первые результаты которых были опубликованы ранее, показали возможность применения сплайн-аппроксимации в проектировании трасс линейных сооружений, несмотря на принципиальные отличия используемых сплайнов от рассматриваемых в теории и ее приложениях. Главное отличие состоит в том, что в проектировании трасс нельзя заранее считать известным число элементов сплайна. Кроме того, в отличие от получивших широкое распространение полиномиальных сплайнов, повторяющимся элементом является связка «отрезок прямой + + дуга окружности» или «отрезок прямой + дуга клотоиды + дуга окружности + дуга клотоиды». Ранее была предложена двухэтапная схема: определение числа элементов искомого сплайна, затем – оптимизация его параметров. Алгоритм решения задачи применительно к проектированию продольного профиля реализован и опубликован. Но этот алгоритм непригоден для проектирования плана трассы, т.к. план трассы, в отличие от профиля, в общем случае является многозначной функцией. Цель работы – обобщить алгоритм на случай сплайн-аппроксимации многозначных функций с учетом особенностей проектирования трасс линейных сооружений.

Методы. На первом этапе используется новая математическая модель, позволяющая применить метод динамического программирования с учетом ограничений на параметры искомого сплайна. На втором этапе используется нелинейное программирование. При этом удается вычислять аналитически производные целевой функции по параметрам сплайна при отсутствии ее аналитического выражения через эти параметры.

Результаты. Разработаны алгоритм аппроксимации многозначных функций, заданных дискретным рядом точек, сплайном, состоящим из дуг окружностей, сопрягаемых отрезками прямых, для решения задачи на первом этапе и алгоритм нелинейного программирования для оптимизации параметров полученного сплайна как начального приближения. В настоящей статье рассматривается только первый этап, т.к. сложный алгоритм второго этапа и его обоснование требуют отдельного рассмотрения.

Выводы. Двухэтапная схема сплайн-аппроксимации при неизвестном числе элементов сплайна пригодна и для аппроксимации многозначных функций, заданных последовательностью точек на плоскости, в частности для проектирования плана трасс линейных сооружений.

Ключевые слова: трасса, план, продольный профиль, сплайн, динамическое программирование, целевая функция, ограничения

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Авторы заявляют об отсутствии конфликта интересов.

INTRODUCTION

The present paper comprises a continuation of the work [1] in which the problem of approximating functions given by a sequence of points on a plane by the spline of special form was considered. There, arcs of circles conjugated by line segments comprised spline elements. The problem was analyzed in relation to the design of the longitudinal profile of linear structures (railways, highways, pipelines for various purposes, etc.). Since a route comprises a plane curve whose plan is its projection on a plane *XOY*, while the longitudinal profile is the function $Z(s)$, where s is the arc length from a given origin in the plan; then, the longitudinal profile comprises a flat curve representing a graph of a single-valued function. The algorithm for spline approximation discussed in [1] is based on this circumstance. Here, the route plan may or may not be the graph of a single-valued function. However, the earlier implemented algorithms turn out to be generally unsuitable for multivalued functions. Thus, other mathematical models and methods are required.

The theory of splines, which appeared in the late 1960s, had been initially considered as a problem of interpolating given points (nodes) of some curve consisting of elements of the same given kind, which would have a common ordinate—and, as a rule, a common tangent—at the spline nodes [2].

From this point on, only the abscissas of nodes needed to be recorded, and researchers became to solve spline approximation problems instead of using interpolation. Then, spline approximation problems began to be solved when varying not only the ordinates but also the abscissas of nodes. In this case, the number of spline elements was taken as known. The most commonly used splines were polynomial and, in particular, cubic [3].

Spline approximation problems arising when designing railway and highway routes and other linear structures differ in that the repeating spline elements comprise groups of elements. When designing the route plan, it is the "straight line + clothoid + circle + + clothoid", etc. Finding the number of spline elements is a separate and rather complex task, as is optimizing the spline parameters that determine its position on a plane.

As noted by Professor Hao Pu in [4], China currently has more than 120 000 km of operating railways, with about 20 000 km of existing railways to be reconstructed by 2025. It is noted that Chinese design engineers are very interested in the emergence of an automatic and accurate method for designing route plans.

When designing the Baikal-Amur Mainline (BAM) in the USSR in the 1970s, the first longitudinal profile design programs were used in all three BESM-4 computers available at the design institutes of the Ministry of Transport Construction in Moscow, Leningrad, and Novosibirsk [5]. Due to the extremely limited technical capabilities of the best computer available at that time (4 096 random access memory cells and 40 000 floating-point operations per second), the absence of visualization tools, and difficulties in input of initial data (punched cards), no significant cost reduction in design was achieved. However, the results obtained at various sections of the BAM proved the efficiency of applying mathematical optimization methods, which was above all due to the improved quality of design solutions [5]. In the 1980s, the domestic system of computer-aided design (CAD) of new railways, which used design programs but without visualization of initial data and results, was developed on ES EVM (the Unified System of Electronic Computers) computers. Consequently, imported systems having such tools and programs but without using optimization methods became widespread during the transition to personal computers. Since then, despite the establishment of a myth that optimization is unnecessary due to designers obtaining optimal solutions interactively, authors such as Hao Pu have shown that this is far from being the case.

At present, the problem of optimizing spline parameters is solved interactively in existing CAD systems[1, 2, 3, 4, 5], with the designer specifying information that uniquely determines the desired design spline. This is essentially the method of element selection in graphics mode with visual control: the computer is used in place of template and ruler without the application of mathematical optimization methods. Therefore, the quality of the results depends on the experience, intuition, and motivation of the designer. Moreover, such "screen crawling" is a rather labor-intensive process. This would seem to justify research on formalizing the problem in mathematical models and applying mathematically correct optimization algorithms. However, in its place, various heuristic algorithms have been proposed both in Russia and abroad. The given points are connected by line segments to obtain a broken line (first-order spline) that must be replaced by a spline with circles conjugated by lines or clothoids and lines at the smallest (in a certain sense) deviation from the original spline (broken line). At the same time, technical constraints have been imposed on the desired spline parameters to ensure normal operation of the designed new or reconstructed structure.

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[1] Bentley Rail Track. URL: https://www.bentley.com/-media/1EA2B937CB5B42BEA5EAE802620C0BA3.ashx. Accessed January 15, 2022.

[2] CARD/1. URL: http://card-1.ru/. Accessed January 15, 2022 (in Russ.).

[3] Autodesk. URL: https://www.architect-design.ru/autodesk/autocad/. Accessed January 15, 2022 (in Russ.).

[4] Topomatic Robur. URL: http://www.topomatic.ru/. Accessed January 15, 2022 (in Russ.).

[5] Credo-Dialog. URL: https://credo-dialogue.ru/. Accessed January 15, 2022 (in Russ.).

The first studies on designing a route plan considered curvature plots [6–8] used as a basis to determine straight inserts. However, this idea has not been developed further due to the extreme difficulty of obtaining, even visually, a straight insert of 30–35 m length on the disrupted route; this is especially true when the curves of the same sign are conjugated in complex cases of surveying points every 20 m.

Next, software developers moved towards the construction of angle diagrams [9], i.e., graphs of the angle of the current polyline element with the *OX* axis against the distance (polyline length) from the starting point. In such graphs, the straight line, circle, and clothoid correspond to the horizontal line, sloping line, and second-degree parabola in the route plan, respectively. The task then becomes to determine element boundaries and perform spline calculations.

Here, the recent work on automated designing the route plan of reconstructed railways by Hao Pu et al. [4] should be noted. The paper deals with the analysis of studies in this field concluding that existing methods do not allow the problem to be solved automatically but are only capable of generating a local optimal solution with allowance for several constraints. In addition, it is noted in [4] that automatic determination of the number of curves, lengths of circles, clothoids, and straight inserts is a complex task. For this reason, it is proposed to find the number of spline elements (circular curves with no allowance for the presence of clothoids) at the first stage using a heuristic algorithm with further result optimization using genetic algorithms [10–19]. According to [4], after preliminarily approximating the boundaries of the straight line by the angle diagram, a heuristic algorithm called a “swing iteration” is proposed for reclassifying point location and determining the position of straight lines more precisely, along with subsequent circular and transitive curves. In a swing iteration, the segment boundary of a geometric element is repeatedly changed from left to right, then right to left, and finally stabilized. It follows from [4] that genetic algorithms have allowed significant improvements to the first stage result while solving the real problem.

An apparently more reliable approach utilizes the same two-stage scheme for solving the problem, but with mathematically correct algorithms: dynamic programming algorithms for determining the number of elements and their parameter approximates at the first stage and nonlinear programming algorithms for optimizing the obtained spline parameters at the second stage. This scheme has been successfully used in designing the longitudinal profiles of railways and highways. When designing railways, the spline in the form of a broken line was originally used [20]; when moving to the design of highways, a spline consisting of vertical circular curves conjugated by straight lines was used [1]. For designing the longitudinal profiles of roads, a spline having elements of second-degree parabolas has also been used [21].

The problem of spline approximation of multivalued functions is relevant for computer-aided design of a route plan, which in general comprises a graph of precisely this function.

Spline becomes a multivalued function not only in the presence of curves having tangent line angles with the *OX* axis greater than 90° but also in the case of several curves of the same sign having small rotation angles, but at a large total angle of rotation. In general, it is also necessary to consider curves with rotation angles greater than 180°.

In this paper, we present dynamic programming features for solving this problem. First, we consider spline consisting of arcs of circles conjugated by straight lines as a multivalued function. This is a separate problem, since variable curvature curves that include clothoids are not used when designing route plans of some linear structures, for example, pipeline trenches of different purposes. This much simpler problem requires significantly less computation at the first stage than when using a spline with clothoids. In addition, when using clothoids of short lengths and large circular curve radii their insertion results in insignificant shifts of the resulting spline with circles, since deviation p of the circle of radius R from the angle side to which it fits with a clothoid of length l into may be calculated by the well-known formula $p = l^2/(24R)$. Thus, $p < 0.08$ m is satisfied at $l = 30$ m and $R = 500$ m.

Therefore, a spline with circles may be considered generally as the initial approximation for the second stage. In any case, the number of curves is not further changed; the first stage may be repeated at a known number of elements to find a spline with clothoids.

This drastically reduces the number of calculations with the use of dynamic programming, since it is not necessary to consider replacing two curves by one at a known number of elements.

Optimizing parameters of the spline as a multivalued function using nonlinear programming is a complex problem with solution to be discussed in a separate paper.

PROBLEM STATEMENT AND FORMALIZATION

For a given sequence of points on a plane (Fig. 1), we shall obtain a spline consisting of arcs of circles conjugated by line segments, whose parameters satisfy the constraint system, while the sum of deviation squares of given points from the spline is minimal. If there are areas where it is necessary to obtain small deviations, the weighted sum of squares can be used instead of the simple solution. In addition, the constraints in the form of inequalities can be imposed on deviations at separate points. Unfortunately, it is impossible to fix the point within the discrete search at this stage.

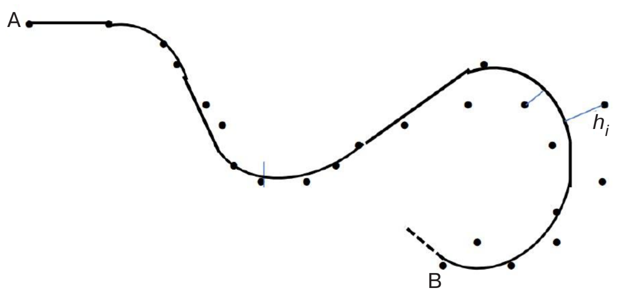

Fig. 1. Starting points and approximating spline

The starting point A and end point B are set along with their directions and not changed during the spline search. These may or may not coincide with origins.

The deviations are calculated for normal to spline. If the number of points is n and their deviations from the spline are h_i (i = 1, 2, ..., n), then the sum $\sum_{i=1}^{n} h_i^2$ must be minimal, subject to the following constraints on spline parameters: the lengths of line segments and arcs of circles must not be less than the specified values, while the radii of circular curves must be within the specified limits.

At the first stage, it would be convenient to consider the elements in the following order: curve + straight line, etc. If the number of these links is k, the line lengths are L_j^{sl}, the curve lengths are L_j^{c}, and the radii are R_j, then the constraints on spline parameters may be formalized by the following system of inequalities:

$$L_j^{\text{sl}} \geq L_{\min}^{\text{sl}}, \quad (1)$$

$$L_j^{\text{c}} \geq L_{\min}^{\text{c}}, \quad (2)$$

$$R_{\min} \leq |R_j| \leq R_{\max} \ (j = 1, 2, \ldots k). \quad (3)$$

The radii of the curves are positive when moving counterclockwise and negative otherwise. All limit values $L_{\min}^{\text{sl}}$, $L_{\min}^{\text{c}}$, $R_{\min}$, and $R_{\max}$ are given.

Clearly, it would be sufficient to find the coordinates of each curve origin and the tangent direction in it. The first curve origin is considered given. This may be point A (Fig. 1) or another point on the tangent drawn from point A. However, if the initial line length is considered unknown at this stage, then the problem becomes much more complex, as can be seen below. However, it is possible to avoid significant complications by specifying several possible points for the first curve origin from which the initial line length may be consequently derived. The same procedure may be carried out with the end point along with the specification of several starting and ending directions.

The basic concept in dynamic programming—"system state"—may be defined as an aggregate of the starting point of the next curve and the tangent direction to the curve at this point. To that end, normals to the given polyline should be constructed at starting points. These comprise a line connecting a given point to the center of a circle drawn through three adjacent points if they are not on the same line, or a normal to this line (Fig. 2). However, it is not necessary to construct normals at the beginning and end of the route in sections of length $L_{\min} = L_{\min}^{\text{sl}} + L_{\min}^{\text{c}}$ from the starting and end points, respectively, since the desired points of curve origins cannot in any case be located in these sections due to constraints (1)–(3).

Since the initial direction is given, moving from the beginning to the end at each point, the direction of the external normal—and, respectively, the tangent—may be determined so that they constitute the right-hand triple. The angles of external normals with the OX axis (γ_j in Fig. 2) are precalculated. The tangent direction is determined by angle ($\gamma_j - \pi/2$) with the OX axis. The starting point coordinates along with the tangent direction determine one "system state" on each normal. Since the curve origin does not necessarily coincide with the starting point, several points on each normal with step Δ (Fig. 2) and several possible tangent directions at each point on each normal (the angle side which the circle fits into) may be set.

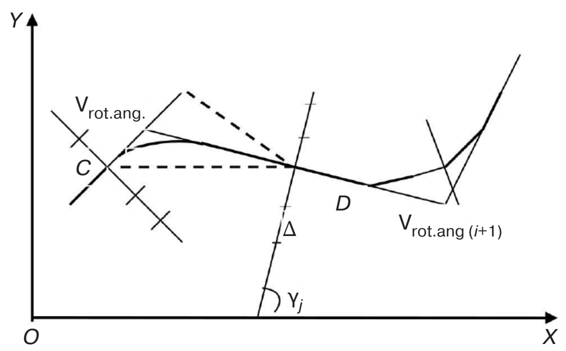

Fig. 2. Defining the normals and sets of "system states" $V_{\text{rot.ang.}}$ is the rotation angle vertex

In this way, a set of possible states may be constructed. The process of obtaining the spline is reduced to a dynamic programming problem: construct a path (sequence of states) to transfer the "system" from the initial state to the final one with minimum costs (at the minimum of the objective function). Sequential states should be selected with allowance for constraints (1)–(3) and constraints on displacements at separate points, if given.

CONSTRUCTING A SPLINE USING THE DYNAMIC PROGRAMMING METHOD

When constructing a path from the initial to the final point in accordance with R. Bellman's optimality principle [22], options for achieving the same state by different paths allowed by constraints are considered and compared; eventually, one option with a smaller value of the objective function is left in each state.

Implementing this rule requires setting some more parameters: the already mentioned discretes by normals Δ and angles φ, as well as their numbers per normal; *maxrix* is the maximum allowed deviation of the spline from surveying points, while L_{max} is the maximum length of the link "circle + straight line," i.e., the maximum distance (by initial broken line) between two subsequent states (curve origins). Typically, $L_{max} = (3–4)L_{min}$, but it may be greater in the presence of long curves. Due to the simplicity of the algorithm (in terms of low computational resources), it is reasonable to set L_{max} "with a margin" and limit the link length, if L_{max} already contains two curves with different sign, since these curves cannot be replaced by a single link with admissible deviations. The *maxrix* value, which specifies the search range on a plane with respect to the original polyline, should also be set with care when analyzing specific data. If small values are set, there may be no solution in the corresponding area due to constraints. Setting large *maxrix* values does not affect the search accuracy but results in the increased amount of computation that is not very significant in this case.

First stage of the algorithm

For the starting point A (if there are several starting points given, then for each of them sequentially), the normals in the range from L_{min} to L_{max} (points C and D in Fig. 3) are considered. For every point on every normal and every tangent direction at this point, the corresponding vertex of the angle of rotation at the intersection with the initial direction is determined. These comprise the points V_1 and V_2 in Fig. 3. Some other directions are shown as dashed lines.

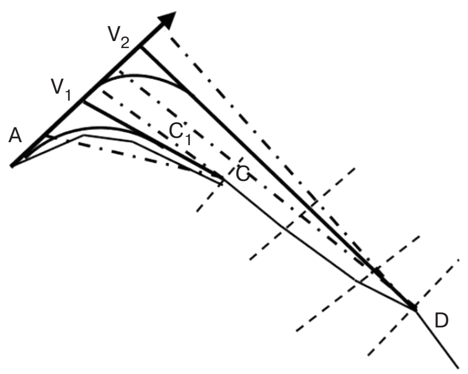

Fig. 3. Option construction at the first stage of the algorithm

The distances from each angle vertex to the starting point and to the point on the normal may be obtained as follows. For the first vertex, the distances are AV_1 and V_1C, respectively. If $AV_1 > V_1C_1 - L^{sl}_{min}$, then this option of selecting starting points is rejected. Otherwise, point C_1 on the angle side V_1C is found such that $AV_1 = V_1C_1$

The distance CC_1 is the length of the straight line in the desired link "circle + straight line."

$$AV_1 = R\mathrm{tg}(\alpha/2),$$

where α is the rotation angle, i.e., the difference between the angles of the angle sides and the *OX* axis. From here, value *R*, followed by the center of the circle using point A and the normal to the initial direction, may be obtained. Since a search of the radius values is not required, the starting point of the curve has also been fixed, along with several options to be used if finding it is not possible. When constraint (3) (on radius) or constraint (2) (on curve length) is not satisfied, the next state may be considered. Here, it should be noted that if the constraints are violated, many states could be excluded from consideration.

If constraints (1)–(3) are satisfied, then distances h_i to the arc of the circle (before going beyond the arc) and then distances to the straight insert for remaining points are found. Should $h_i > maxrix$ or the constraints on displacement of some points be violated, the remaining distances are not calculated, while this option of locating starting points of the curve is rejected. Otherwise, the objective function value is calculated and memorized along with all data required to subsequently restore the spline (radius, coordinates of arc end C_1, etc.).

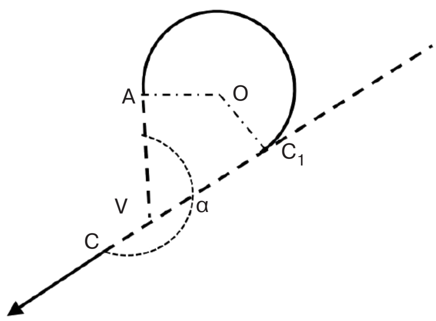

Fig. 4. Calculation at angles of rotation greater than π

If rotation angle $\alpha > \pi$ (Fig. 4), no particular difficulties arise. In this case, $AV = R|\mathrm{tg}(\pi - \alpha/2)| = R|\mathrm{tg}(\alpha/2)|$, CC_1 is straight insert, while the arc length $L = R\alpha$. Hence, checking constraints (1)–(3) is performed in the same way as for small rotation angles. In the theoretically possible case of $\alpha = \pi$ (Fig. 5), there is no angle vortex, the radius is half the distance between parallel lines; the arc length $L = \pi R$. Straight insert CC_1 as well as values *R* and *L* may be unacceptable. Deviations from survey points are calculated in the same way as above.

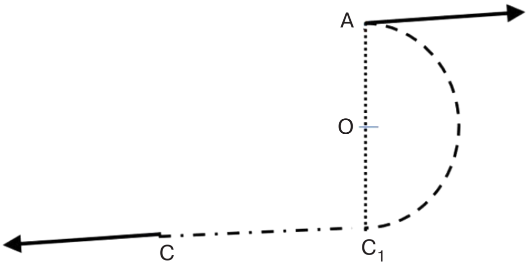

Fig. 5. Calculation when parallel lines

It should be noted that there is no option comparison and their rejection by the objective function value at the first stage.

General stage of the algorithm

At next stages, all normals ranging from $2L_{min}$ to S is L_{min}, where S is the length of the original polyline are considered. For each such normal, the possibility of constructing the link "circle + straight line" with the starting point belonging to the preceding normal that is at least L_{min} and at most L_{max} away from the considered normal is analyzed. The difference from the first stage of the algorithm is that now there are many options of the left angle side instead of one for each point and direction on the considered normal. All acceptable options of transition to the considered state from previous states are compared by the objective function and the best one memorized. The corresponding state of the constructed link origin is also memorized. In other words, transitions not only unacceptable by constraints but also ineffective are rejected. It may turn out that two admissible paths consisting of different number of links result in the same state. This means that dynamic programming considers paths with different number of elements to eventually determine the number of elements for the approximating spline, having previously been unknown.

At the last stage, the same actions are performed for the final state or several such states, if given. As a result, the comparison yields the objective function value for the best option of the path. The path, i.e., the desired spline, is defined by the usual reversal from the found final best state along the chain of links used in dynamic programming, since the corresponding curve origin has been memorized for each state [20, 21].

REDUCING THE SEARCH OF OPTIONS

The considered problem and the dynamic programming algorithm may be characterized by the presence of constraints not only avoiding an unnecessary complication of the problem but also allowing the search of options to be significantly reduced. When the attempt to construct the link "arc circle + line segment" is unsuccessful, many transitions to another state (displacements along the right normal or rotations of the tangent, Fig. 3) are obviously inefficient due to the constraints remaining violated during this transition. For example, the minimum radius constraint is violated when fitting into the angle. Obviously, with the rotation angle α unchanged, the tangent T (the distance from the tangent point to the angle vertex) cannot be reduced since $R = T/\text{tg}(\alpha/2)$. Accordingly, displacements along the right normal reducing T are unacceptable.

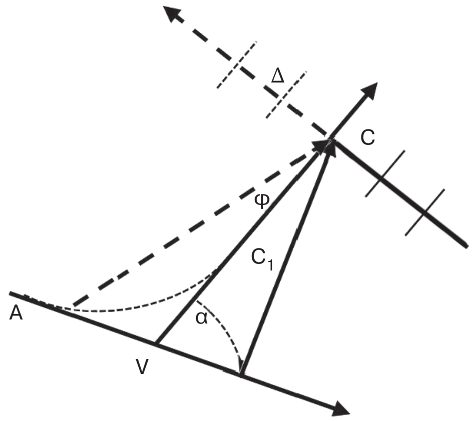

Fig. 6. Rejecting transitions when constraints are violated

In Fig. 6, T = AV. Transitions to the new state from point C (displacements along the normal and rotations of the tangent) unnecessary $R < R_{min}$ are shown dashed (e.g., to the left of point C). If straight insert C_1C is less than the acceptable one, the constraints on radii and curve lengths are satisfied; then, conversely, only dashed displacements and rotations result in its increase. However, due to the reduction in tangent AV and curve length, the "margin" for these parameters may be insufficient.

At any arrangement of normals and different rotation angles, each violated constraint provides information concerning which states cannot be efficiently transferred to on the given normal. Although this analysis slightly complicates the algorithm for spline construction, it is fully compensated by reducing the search of options.

CONCLUSIONS

The method for approximating the sequence of points on a plane by a spline with circles and straight lines described in this paper may not only be used in designing routes of linear structures but also in processing generally any type of data when constructing paths. As for its practical application in designing routes of linear structures, the question remains open and is the lack of interested users.

In Russia, the number of design works may increase in future due to new railway and highway construction and reconstruction projects. This may result in the growing interest of relevant authorities in reducing construction costs by optimizing design solutions, thus giving relevance to developing new design algorithms and programs.

As was the case during the Soviet period, the development of such design approaches requires theoretical and experimental research by specialized scientific departments. The first Russian developments in optimization of design solutions were significantly ahead of their foreign equivalents. However, foreign authors even now propose mainly various heuristic algorithms without using modern mathematical achievements. The present paper and its sequel dealing with the optimization of splines with arcs of circles and straight lines for the approximation of multivalued functions using nonlinear programming paves the way for a solution of the more complex and important—in theoretical and practical terms—problem of approximating multivalued functions using composite splines with clothoids. This is a relevant topic for further research in this field.

Authors' contribution. All authors equally contributed to the research work.

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