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RESEARCH ARTICLE

New operational relations for mathematical models of local nonequilibrium heat transfer

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Abstract

Objectives. Recently, interest in studying local nonequilibrium processes has increased in the context of the development of laser technologies, the possibility of reaching ultrahigh temperatures and pressures, and the need for a mathematical description of various physical processes under extreme conditions. In simulating local nonequilibrium processes, it becomes necessary to take into account the internal structure of investigation subjects, which significantly complicates the classical transport models. An important stage here is to construct mathematical models of various physical fields in which their spatiotemporal nonlocality should be taken into account. For these purposes, hyperbolic equations are used for a wide class of phenomena and, first of all, for unsteady-state heat conduction processes based on the generalized Maxwell–Cattaneo–Luikov–Vernotte phenomenology. Mathematical models in the form of boundary value problems for hyperbolic equations are called generalized boundary value problems. These problems differ significantly in solving difficulty from the classical ones based on Fourier phenomenology. The specificity of these problems is the relative simplicity of the initial mathematical models, together with the difficulty of solving them in an analytically closed form. Hence, very little success has been achieved in finding exact analytical solutions to problems of this kind. The most acceptable approach to solving them is operational calculus. However, it gives analytical solutions in the Laplace transform space as complex functional structures, the inverse transforms of which are not available in well-known reference books on operational calculus. On this path, serious computational difficulties arise. The study aimed to analyze a set of nonstandard transforms arising from the operational solution of mathematical models of local nonequilibrium heat transfer and to obtain their inverse transforms.

Methods. Methods and theorems of operational calculus, methods of contour integration of complex transforms, and the theory of special functions were used.

Results. Operational calculus was developed for mathematical models of local nonequilibrium heat transfer in terms of the theory of unsteady-state heat conduction for hyperbolic equations (wave equations). Nonstandard operational transforms, the inverse transforms of which are unavailable in the literature, were considered. It was shown that the presented transforms are common to operational solutions of a wide class of generalized boundary value problems for hyperbolic equations in the theory of heat conduction, diffusion, hydrodynamics, vibrations, propagation of electricity, thermomechanics, and other areas of science and technology. Partially bounded and finite domains were explored. Illustrative examples were given, namely, the results of numerical experimental studies of a local nonequilibrium heat transfer process that took into account the finiteness of the heat transfer rate, which had a wave character. The latter was expressed by the presence of the Heaviside step function in the analytical solution of the problem. The physical meaning of the finiteness of the heat transfer rate was substantiated. The isochron was constructed for the temperature function in a partially bounded domain. It was shown that the temperature profile has a discontinuity on the surface of the propagating wave front. This leads to the retention of heat outflow beyond the discontinuity boundary. This is a characteristic feature of the analytical solutions of the wave equations, along with the possibility to describe the analytical solution of the problem as equivalent integral relations, which noticeably simplify numerical calculations.

Conclusions. The inverse transforms of nonstandard operational (Laplace) transforms were presented, which are contained in the operational solutions of a wide class of problems of local nonequilibrium (heat, mass, momentum) transfer processes, electrical circuits, hydrodynamics, oscillation theory, thermomechanics, and others. Illustrative examples were given, and the possibility of transition from one form of an analytical solution to another equivalent form was shown. The presented analytical solutions of hyperbolic heat transfer models in canonical domains are new in classical thermal physics.

Keywords: nonstandard operational transforms, inverse transforms, mathematical models of local nonequilibrium heat transfer, analytical solutions

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НАУЧНАЯ СТАТЬЯ

Новые операционные соотношения для математических моделей локально-неравновесного теплообмена

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Резюме
Цели. В последние годы усилился интерес к изучению локально-неравновесных процессов в связи с развитием лазерных технологий, возможностью получения сверхвысоких температур и давлений, а также ввиду необходимости математического описания различных физических процессов в экстремальных условиях. При моделировании локально-неравновесных процессов возникает необходимость учета внутренней структуры исследуемых объектов, что приводит к существенному усложнению классических моделей переноса. Важным этапом в развитии указанной области является построение математических моделей разнообразных физических полей с учетом их пространственно-временной нелокальности. Для этих целей используются уравнения гиперболического типа для широкого класса явлений и, прежде всего, для процессов нестационарной теплопроводности на основе обобщенной феноменологии Максвелла – Каттанео – Лыкова – Вернотта. Математические модели в виде краевых задач для уравнений гиперболического типа носят название краевых задач обобщенного типа. Эти задачи значительно отличаются от классических на основе феноменологии Фурье по сложности их решения. Их специфика заключается в относительной простоте исходных математических моделей и трудности решения в аналитически замкнутом виде. Отсюда весьма незначительные успехи в нахождении точных аналитических решений такого рода задач. Наиболее приемлемый метод их решения – операционный, но он приводит к аналитическим решениям в пространстве изображений по Лапласу в виде сложных функциональных конструкций, оригиналы которых не содержатся в известных справочниках по операционному исчислению. На этом пути возникают серьезные трудности вычислительного характера. Цель работы – рассмотреть серию нестандартных изображений, возникающих при операционном решении математических моделей локально-неравновесного теплообмена и получить их оригиналы.

Методы. Использованы методы и теоремы операционного исчисления, методы контурного интегрирования сложных изображений, теория специальных функций.
Результаты. Представлено развитие операционного исчисления для математических моделей локально-неравновесного теплопереноса в терминах теории нестационарной теплопроводности для уравнений гиперболического типа (волновых уравнений). Рассмотрены нестандартные операционные изображения, оригиналы которых ранее были неизвестны. Показано, что приведенные изображения являются характерными для операционных решений широкого класса обобщенных краевых задач для уравнений гиперболического типа в теории теплопроводности, диффузии, гидродинамики, колебаний, распространения электричества, термомеханики и других направлений науки и техники. Изучены частично ограниченные и конечные области. Приведены иллюстративные примеры в качестве численных экспериментов локально-неравновесного процесса теплообмена с учетом конечной скорости распространения теплоты, имеющей волновой характер. Последнее выражается наличием ступенчатой функции Хевисайда в аналитическом решении задачи. Обоснован физический смысл конечной скорости распространения теплоты; построена изохрона для температурной функции в частично ограниченной области и показано, что на поверхности фронта идущей волны температурный профиль имеет разрыв. Это приводит к задержанию оттока теплоты за границу разрыва – характерная особенность аналитических решений волновых уравнений, к которой следует добавить возможность описания аналитического решения задачи в виде эквивалентных интегральных соотношений, существенно упрощающих числовые расчеты.
Выводы. Представлены оригиналы нестандартных операционных изображений (по Лапласу), входящие в операционные решения широкого класса задач локально-неравновесных процессов переноса (теплоты, массы, импульса), электрических цепей, гидродинамики, теории колебаний, термомеханики и других областей. Приведены иллюстративные примеры и показана возможность перехода от одной формы аналитического решения к другой эквивалентной форме. Представленные аналитические решения гиперболических моделей теплопереноса в областях канонического типа являются новыми в классической теплофизике.

Ключевые слова: нестандартные операционные изображения, оригиналы, математические модели локально-неравновесного теплопереноса, аналитические решения

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INTRODUCTION

The classical models of the analytical theory of heat transfer originated from Fourier's linear gradient relation

$$\overline{q}(M,t) = -\lambda gradT(M,t),$$

presented by Fourier in his *Mémoire sur la propagation de la Chaleur dans les corps solides* (On the Propagation of Heat in Solid Bodies) in Paris in 1807. Fourier finalized his theory in 1822 in the essay *Théorie analytique de la chaleur eur* (The Analytical Theory of Heat), which Kelvin called "the great mathematical model" [1]. Together with the energy equation for isotropic solids,

$$c\rho\frac{\partial T(M,t)}{\partial t} = -\mathrm{div}\overline{q}(M,t) + F(M,t),$$

Fourier's law leads to parabolic unsteady-state heat transfer equation

$$\frac{\partial T(M,t)}{\partial t} = a\Delta T(M,t) + \frac{1}{c\rho}F(M,t),\ M \in D,\ t > 0, \quad (1)$$

and to boundary value problems for equation (1) under initial condition

$$T(M,t)\big|_{t=0} = \Phi_0(M), M \in \overline{D} \quad (2)$$

and boundary conditions

$$\beta_1\frac{\partial T(M,t)}{\partial n} + \beta_2 T(M,t) = \beta_3\varphi(M,t),\ M \in S,\ t > 0. \quad (3)$$

Here, D is a finite of partly bounded convex domain of $M(x, y, z)$, S is a piecewise smooth surface bounding the domain D, and $\overline{n}$ is the external normal to S (a vector, continuous at points of S). The parameters of problem (1)–(3) are thermophysical characteristics of the medium, which are constant in a range of temperatures within the transition points [2]. Some paradoxes of using model concepts (1)–(3) were repeatedly noted. One of them is the absence of inertia of the process of heat conduction in Fourier's law because of the ignorance of the mechanism of heat transfer by constituent particles of matter (electrons, molecules, ionic lattices) and the relaxation time related to the mean free time of microparticles. A consequence of this is the conclusion of the infinity of the heat transfer rate, which follows from the analytical solutions of models (1)–(3). Another paradox is the singularity of the heat flow and the velocity of motion of isotherms at $x > 0$ and $t > 0$ as $x \to 0$ ($t \to 0$). Nonetheless, these circumstances do not limit the applicability range of mathematical models (1)–(3), which encompasses more and more substantive subjects of investigation and a wider and wider variety of applications [3–7].

Recently, the interest in studying local nonequilibrium processes has increased in view of the wide potential of their practical application [8–14], namely, creation of new technologies to obtain nanomaterials and coatings with unique physicochemical properties (binary and multicomponent metal alloys, polymer materials, amorphous metal semiconductors, nanoliquids, and colloidal and bio- and cryosystems); optimization of laser application conditions; intense heating and cooling of nanoelectronical and nanotechnical components; and heating, melting, and ablation of matter on exposure to ultrashort laser pulses; etc. To describe the intensification of thermal processes under these conditions, it was required to refine Fourier's hypothesis. This was done by taking into account the local nonequilibrium using relation

$$\bar{q}(M,t) = -\lambda \operatorname{grad} T(M,t) - \tau_{\mathrm{r}} \frac{\partial \bar{q}(M,t)}{\partial t}, \quad (4)$$

which implies the conclusion of the finiteness of the heat transfer rate. Here, the time τ_{r} is a measure of the inertia of the heat flow, which is related to the heat transfer rate by the expression $v_{\mathrm{T}} = \sqrt{a/\tau_{\mathrm{r}}}$. The necessity to take into account the effect of the finiteness of the heat (mass) transfer rate was indicated by Maxwell in gas dynamics theory [15], Luikov in investigation of the heat and moisture transfer in capillary porous bodies [16], and Cattaneo [17] and Vernotte [18] in heat conduction theory. The energy equation and relation (4) lead to hyperbolic heat transfer equation

$$\frac{\partial T(M,t)}{\partial t} = a\Delta T(M,t) - \\ -\tau_{\mathrm{r}} \frac{\partial^2 T(M,t)}{\partial t^2} + \frac{\tau_{\mathrm{r}}}{c\rho}\left[\frac{\partial F(M,t)}{\partial t} + \frac{1}{\tau_{\mathrm{r}}} F(M,t)\right] \quad (5)$$

and to the corresponding generalized boundary value problems of unsteady-state heat conduction. In the mathematical formulation of these problems, the corresponding local nonequilibrium boundary conditions should be used. Using standard local equilibrium boundary conditions (3) (which is quite frequently done in publications on analytical thermal physics) can lead to physically contradictory results (e.g., to negative values of absolute temperature [19]). These issues were considered in detail previously [20]. Correct generalized boundary conditions were formulated based on relation (4) in the integral and equivalent differential forms. For example, for thermal heating (cooling) conditions, the Neumann boundary condition has the form

$$\frac{1}{\tau_r}\int_0^t \frac{\partial T(M,\tau)}{\partial n}\Big|_{M\in S} \exp\left(-\frac{t-\tau}{\tau_r}\right) d\tau = \\ = \pm(q_0/\lambda) S_+(t), t \ge 0, \quad (6)$$

and in the case of heating by the medium, one should write

$$\frac{1}{\tau_{\mathrm{r}}}\int_0^t \frac{\partial T(M,\tau)}{\partial n}\Big|_{M\in S} \exp\left(-\frac{t-\tau}{\tau_{\mathrm{r}}}\right) d\tau = \\ = h\{T(M,t)|_{M\in S} - \left[T_0 + S_+(t)(T_c - T_0)\right]\}, \; t \ge 0. \quad (7)$$

Here,

$$S_+(t) = \begin{cases} 1, \; t > 0, \\ 0, \; t \le 0. \end{cases} \quad (8)$$

As $(1/h) \to 0$ ($h \to \infty$), from relation (7), the boundary condition during thermal heating is obtained:

$$T(M,t)|_{M\in S} = T_0 + S_+(t)(T_c - T_0), \; t \ge 0. \quad (9)$$

Note that hyperbolic equation (5) for local nonequilibrium heat and mass transfer processes was obtained for the first time by Fock [21] and Davydov [22] under the assumption of the finiteness of the velocity of energy- or mass-transferring particles. Equation (5) was also derived by Predvoditelev [23] by analyzing the velocities of isothermal surfaces using Riemann's concepts, i.e., after completely abandoning the use of relaxation formula (4).

NEW OPERATIONAL RELATIONS FOR HYPERBOLIC MATHEMATICAL MODELS

Generalized transfer problems for Eq. (5) differ significantly from classical problems (1)–(3) because analytical solutions of the former are more difficult to obtain. The specificity of these problems is the relative simplicity of the initial mathematical models together with the difficulty of solving them in an analytically closed form. Hence, very little success was achieved in finding their exact analytical solutions, which were mostly found within partially bounded domains. The main approach to solve these problems is operational calculus, which is, however, accompanied by two main challenges. Whereas solving a problem using operational calculus is not too difficult, the inverse transforms are difficult to obtain because of their absence in operational calculus tables. The formal application of operational calculus theorems to finding the inverse transforms can lead erroneous results because the sought inverse transforms should contain the Heaviside step function [24], the formal appearance of which not always can be ensured. A natural way out of this situation is to develop artificial techniques or a complex transition to the inverse transforms by contour integration of the transforms [24].

Partially bounded domain

Let us consider the operational solution of homogeneous equation (5) in the domain $\Omega = \{M(z, t): z \geq 0, t \geq 0\}$ in the dimensionless variables

$$\xi = v_p z/a,\ \tau = v_p^2 t/a,\ \beta = v_p/v_T,$$
$$W(\xi,\tau) = [T(z,t) - T_0]/(T_c - T_0),$$
$$v_T = \sqrt{a/\tau_r},$$

where $v_p = \sqrt{2G(1-\nu)/[\rho(1-2\nu]}$ is the expansion wave velocity in an elastic medium, G is the shear modulus, ν is Poisson's ratio, and ρ is the density. The problem has the form

$$\frac{\partial W}{\partial \tau} = \frac{\partial^2 W}{\partial \xi^2} - \beta^2 \frac{\partial^2 W}{\partial \tau^2},\ \xi > 0,\ \tau > 0, \quad (10)$$

$$\left.\begin{array}{l} W(\xi,\tau)\Big|_{\tau=0} = \dfrac{\partial W(\xi,\tau)}{\partial \tau}\Big|_{\tau=0} = 0,\ \xi \geq 0, \\ |W(\xi,\tau)| < \infty,\ \xi \geq 0,\ \tau \geq 0 \end{array}\right\}. \quad (11)$$

Let $\overline{W}(\xi,p) = \int_0^\infty \exp(-p\tau) W(\xi,\tau) d\tau$ be the Laplace transform of the function $W(\xi, \tau)$. From Eq. (10), one can find

$$\overline{W}(\xi,p) = \bar{f}(p)\exp\left(-\xi\sqrt{\beta^2 p^2 + p}\right).$$

The transform written for $\overline{W}(\xi,p)$ determines the further purpose of this study, which is to obtain the inverse transforms of a set of transforms of the form $\bar{f}(p)\exp\left[-\xi\sqrt{(p+2\alpha)(p+2\beta)}\right] = \bar{f}(p)\exp[-\xi\bar{\mu}(p)]$, where $\bar{\mu}(p) = \sqrt{(p+2\alpha)(p+2\beta)}$, at various $\bar{f}(p)$ with the Heaviside step function.

Let us emphasize once again the specific features of the further calculations. The pioneering publications on the simplest hyperbolic transfer models in 1970s showed [25, 26] that the heat conduction according to the mathematical models for Eq. (5) is characterized by the wave behavior, which is expressed by the presence of the step function $\eta(\tau - \xi)$ in the analytical solution of Dirichlet boundary value problem (10)–(11) at $W(0, \tau) = 1$, $\tau > 0$. At any time, there is a thermal wake region at $\xi < \tau$ and an unperturbed region at $\xi > \tau$ (Fig. 1) (at points of the region at ξ above $\xi = \tau$, the temperature value remains unchanged and equal to the initial value). On the surface of the front of the propagating wave, we have $\xi = \tau$, and the temperature profile on the front has a discontinuity, the amplitude of which rapidly decreases with increasing heating time. It is in the region behind the front of the heat wave that there is a significant difference between the solutions of hyperbolic equation (5) and parabolic equation (1) (in the latter case, the solutions are smooth and considerably exceed the initial value). It is these features that are explained by the presence of the step function in the analytical solution of the heat problem.

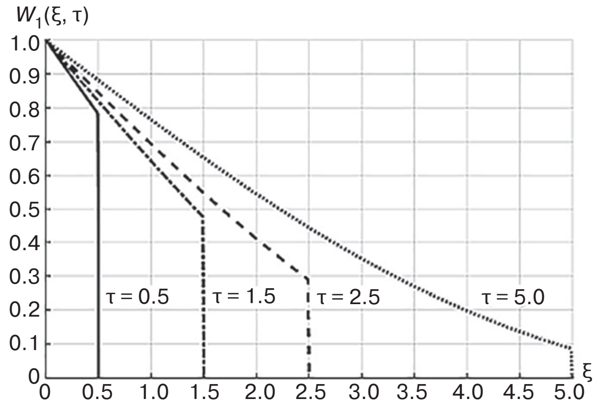

Fig. 1. Isochrone calculated by (29) (β = 1)

Below, only operational calculus relations are considered; therefore, let the time variable be denoted as t (as it is accepted).

Consider the function $\varphi(x) = x^{-m/2} I_m(x^{1/2})$, where $I_m(x^{1/2})$ is the modified Bessel function. In the theory of Bessel functions, there is relation

$$\frac{d^n}{dx^n}[\varphi(x)] = \frac{1}{2^n} x^{-\frac{m+n}{2}} I_{m+n}\left(x^{\frac{1}{2}}\right), \quad (12)$$

which enables oner to write the Maclaurin series for $\varphi(x + x_0)$:

$$\varphi(x + x_0) = (x + x_0)^{-\frac{1}{2}m} I_m\left[(x + x_0)^{\frac{1}{2}}\right] = \sum_{n=0}^{\infty} \frac{x_0^n}{n!} \frac{1}{2^n} x^{-\frac{m+n}{2}} I_{m+n}\left(x^{\frac{1}{2}}\right). \quad (13)$$

Set $x_0 = 2\kappa t$, $x = t^2$, and $m = 0$ in expression (13). Then, we obtain

$$\varphi(t) = I_0\left[\sqrt{t(t + 2\kappa)}\right] = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} I_n(t).$$

Take the Laplace transform using the published tables [27]:

$$\bar{\varphi}(p)=\int_0^\infty \exp(-pt)I_0\left[\sqrt{t(t+2\kappa)}\right]dt=$$
$$=\frac{1}{\sqrt{p^2-1}}\sum_{n=0}^\infty \frac{\kappa^n}{n!}\left(p-\sqrt{p^2-1}\right)^n=$$
$$=\frac{1}{\sqrt{p^2-1}}\sum_{n=0}^\infty \frac{(\kappa p-\kappa\sqrt{p^2-1})^n}{n!}=$$
$$=\frac{1}{\sqrt{p^2-1}}\exp(\kappa p-\kappa\sqrt{p^2-1}).$$

Hence, it follows

$$\bar{\varphi}(p)\exp(-\kappa p)=\frac{1}{\sqrt{p^2-1}}\exp(-\kappa\sqrt{p^2-1})$$

and, further,

$$\frac{1}{\sqrt{p^2-1}}\exp(-\kappa\sqrt{p^2-1})\leftarrow$$
$$\leftarrow \varphi(t-\kappa)\eta(t-\kappa)= \tag{14}$$
$$=I_0(\sqrt{t^2-\kappa^2})\eta(t-\kappa).$$

Let us replace κ by δ, and t by σt in relation (14) and make transformations using the similarity theorem [2]; let us then introduce the notation $\xi=\delta/\sigma$. We find

$$\frac{1}{\sqrt{p^2-\sigma^2}}\exp(-\xi\sqrt{p^2-\sigma^2})\leftarrow \tag{15}$$
$$\leftarrow I_0(\sigma\sqrt{t^2-\xi^2})\eta(t-\xi).$$

Set $p\to p+\rho$ on the left-hand side of relation (15) and apply the shift theorem:

$$\frac{1}{\sqrt{(p+\rho+\sigma)(p+\rho-\sigma)}}\exp\left[-\xi\sqrt{(p+\rho+\sigma)(p+\rho-\sigma)}\right]\leftarrow$$
$$\leftarrow \exp(-\rho t)I_0\left(\sigma\sqrt{t^2-\xi^2}\right)\eta(t-\xi).$$

Denote $p+\sigma=2\alpha$ and $\rho-\sigma=2\beta$, whence $p=\alpha+\beta$ and $\sigma=\alpha-\beta$. Now, we finally arrive at the necessary result:

$$\frac{1}{\sqrt{(p+2\alpha)(p+2\beta)}}\exp\left[-\xi\sqrt{(p+2\alpha)(p+2\beta)}\right]=$$
$$=\frac{1}{\bar{\mu}(p)}\exp\left[-\xi\bar{\mu}(p)\right]\leftarrow \tag{16}$$
$$\leftarrow \exp(-\rho t)I_0(\sigma\sqrt{t^2-\xi^2})\eta(t-\xi).$$

From relation (16), we find

$$\frac{1}{p\bar{\mu}(p)}\exp\left[-\xi\bar{\mu}(p)\right]\leftarrow$$
$$\leftarrow\left[\int_\xi^t \exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-\xi^2})d\tau\right]\eta(t-\xi). \tag{17}$$

Application of the convolution theorem to relation (16) gives

$$\frac{1}{\bar{\mu}(p)}\exp[-\xi\bar{\mu}(p)]\bar{f}(p)\leftarrow$$
$$\leftarrow\left[\int_\xi^t f(t-\tau)\exp(-\rho t)I_0\left(\sigma\sqrt{t^2-\xi^2}\right)d\tau\right]\eta(t-\xi). \tag{18}$$

Differentiation of relation (18) with respect to ξ gives

$$\exp\left[-\xi\bar{\mu}(p)\right]\bar{f}(p)\leftarrow f(t-\xi)\exp(-\rho\xi)+$$
$$+\sigma\xi\int_\xi^t f(t-\tau)\exp(-\rho\tau)\frac{I_1(\sigma\sqrt{\tau^2-\xi^2})}{\sqrt{\tau^2-\xi^2}}d\tau, \tag{19}$$
$$t>\xi, 0, t<\xi.$$

Set $\bar{f}(p)=1$ in relation (19), whence $f(t)=\delta(t)$ is the Dirac delta function, and relation (19) takes the form

$$\exp\left[-\xi\bar{\mu}(p)\right]\leftarrow$$
$$\leftarrow\left[\exp(-\rho\xi)\delta(t-\xi)+\sigma\xi\exp(-\rho t)\frac{I_1(\sigma\sqrt{t^2-\xi^2})}{\sqrt{t^2-\xi^2}}\right]\times \tag{20}$$
$$\times\eta(t-\xi).$$

Let then $\bar{f}(p)=\frac{1}{p}$ in relation (19); then, $f(t)=1$, and the inverse transform has the form

$$\frac{1}{p}\exp\left[-\xi\bar{\mu}(p)\right]\leftarrow$$
$$\leftarrow\left[\exp(-\rho\xi)+\sigma\xi\int_\xi^t\exp(-\rho\tau)\frac{I_1(\sigma\sqrt{\tau^2-\xi^2})}{\sqrt{\tau^2-\xi^2}}d\tau\right]\times \tag{21}$$
$$\times\eta(t-\xi).$$

A number of problems in thermal shock theory [2] lead to the transform

$$\frac{1}{p}\sqrt{\frac{p+2\beta}{p+2\alpha}}\exp\left[-\xi\bar{\mu}(p)\right].$$

Find the inverse transform. From relation (17), we have

$$\frac{1}{p}\frac{\exp\left[-\xi\bar{\mu}(p)\right]}{\sqrt{(p+2\alpha)(p+2\beta)}}\leftarrow$$
$$\leftarrow\int_{\xi}^{t}\exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-\xi^2})d\tau,\ t>\xi. \tag{22}$$

Multiply relation (22) by 2β and add the result term by term with relation (6) to obtain

$$\frac{1}{p}\sqrt{\frac{p+2\beta}{p+2\alpha}}\exp\left[-\xi\bar{\mu}(p)\right]\leftarrow$$
$$\leftarrow\exp(-\rho t)I_0(\sigma\sqrt{t^2-\xi^2})+ \tag{23}$$
$$+2\beta\int_{\xi}^{t}\exp(-\rho\tau)I_0(\sigma\sqrt{\tau^2-\xi^2})d\tau, t>\xi.$$

Transformation of relation (23) using the expressions $(d/dx)I_0(x)=I_1(x)$ and $I_0(0)=1$ gives

$$\frac{1}{p}\sqrt{\frac{p+2\beta}{p+2\alpha}}\exp\left[-\xi\bar{\mu}(p)\right]\leftarrow\exp(-\rho\xi)+$$
$$+\int_{\xi}^{t}\exp(-\rho\tau)\left[\frac{\sigma\tau I_1(\sigma\sqrt{\tau^2-\xi^2})}{\sqrt{\tau^2-\xi^2}}-\sigma I_0(\sigma\sqrt{\tau^2-\xi^2})\right]d\tau, \tag{24}$$
$$t>\xi.$$

Using the above operational calculus relations and theorems, let us write the inverse, important for applications, of the more general transform:

$$\sqrt{\frac{p+2\beta}{p+2\alpha}}\exp\left[-\xi\bar{\mu}(p)\right]\bar{f}(p)\leftarrow f(t-\xi)\exp(-\rho\xi)+$$
$$+\int_{\xi}^{t}f(t-\tau)\exp(-\rho\tau)\left[\frac{\sigma\tau I_1(\sigma\sqrt{\tau^2-\xi^2})}{\sqrt{\tau^2-\xi^2}}-\sigma I_0(\sigma\sqrt{\tau^2-\xi^2})\right]d\tau,$$
$$t>\xi. \tag{25}$$

Expression (25) has a number of practically important special cases at given $f(t)\to\bar{f}(p)$.

The above operational relations have resolved the problem of finding analytical solutions of Eq. (10) under generalized boundary conditions. However, this problem has an interesting continuation, which is the possibility to represent one and the same analytical solution as different functional expressions. It is significant in this case that a certain cumbersomeness of the analytical writing of solutions can be simplified using special transformations leading to new, previously unknown, analytical solutions. Let us demonstrate this by the example of the Dirichlet boundary value problem for Eq. (10) (the boundary conditions are presented above). An analytical solution of the problem based on relation (21) has the form

$$W(\xi,\tau)=$$
$$=\left[\exp\left(-\frac{\xi}{2\beta}\right)+\frac{\xi}{2\beta}\int_{\xi}^{\tau/\beta}\exp\left(-\frac{x}{2\beta}\right)\frac{I_1\left(\frac{1}{2\beta}\sqrt{x^2-\xi^2}\right)}{\sqrt{x^2-\xi^2}}dx\right]\times \tag{26}$$
$$\times\eta(\tau-\xi\beta)=\Psi_1(\xi,\tau)\eta(\tau-\xi\beta).$$

The transform of sought function (26) has the form

$$\overline{W}(\xi,p)=(1/p)\exp\left[-\beta\xi\sqrt{p(p+1/\beta^2)}\right],$$

the inverse transform of which can be written using the Riemann–Mellin integral as follows:

$$\frac{1}{p}\exp\left[-\beta\xi\sqrt{p(p+1/\beta^2)}\right]\leftarrow$$
$$\leftarrow\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{p}\exp\left[p\tau-\beta\xi\sqrt{p(p+1/\beta^2)}\right]dp= \tag{27}$$
$$=\Psi_2(\xi,\tau).$$

The integrand in relation (27) satisfies the conditions of Jordan's lemma [2] and has two points of branching. Calculation of contour integral (27) gives

$$\Psi_2(\xi,\tau)=1-\frac{1}{\pi}\int_0^{1/\beta^2}\exp(-\rho\tau)\frac{\sin\xi\beta\sqrt{\rho\left(\frac{1}{\beta^2}-\rho\right)}}{\rho}d\rho. \tag{28}$$

Let us now show that the analytical solution of the Dirichlet boundary value problem in the form $W(\xi,\tau)=\Psi_1(\xi,\tau)\eta(\tau-\xi\beta)$ are $W(\xi,\tau)=\Psi_2(\xi,\tau)\eta(\tau-\xi\beta)$ equivalent; i.e., $\Psi_1(\xi,\tau)=\Psi_2(\xi,\tau)$. We have

$$\Psi_1(\xi,\tau)=\frac{\partial}{\partial\xi}\left[-\int_{\xi}^{\tau/\beta}\exp\left(-\frac{x}{2\beta}\right)I_0\left(\frac{1}{2\beta}\sqrt{x^2-\xi^2}\right)dx\right].$$

Differentiation of both sides with respect to τ gives

$$\left[\Psi_1\right]_\tau'=\frac{\partial}{\partial\xi}\left[-\frac{1}{\beta}\exp\left(-\frac{\tau}{2\beta^2}\right)I_0\left(\frac{1}{2\beta}\sqrt{\frac{\tau^2}{\beta^2}-\xi^2}\right)\right]=$$
$$=\frac{\partial}{\partial\xi}\left[-\frac{1}{\beta}\exp\left(-\frac{\tau}{2\beta^2}\right)J_0\left(\frac{1}{2\beta^2}\sqrt{(\beta\xi)^2-\tau^2}\right)\right].$$

Let us now use the (quite rare) integral

$$\int_0^a\frac{\exp(-py)}{\sqrt{ay-y^2}}\cos c\sqrt{ay-y^2}dy=$$
$$=\pi\exp\left(-\frac{ap}{2}\right)J_0\left(\frac{a}{2}\sqrt{c^2-p^2}\right).$$

We find

$$\left[\Psi_1(\xi,\tau)\right]'_\tau =$$

$$= \frac{\partial}{\partial\xi}\left[-\frac{1}{\beta\pi}\int_0^{1/\beta^2}\frac{\exp(-\rho\tau)}{\sqrt{\rho\left(\frac{1}{\beta^2}-\rho\right)}}\cos\left(\beta\xi\sqrt{\rho\left(\frac{1}{\beta^2}-\rho\right)}\right)d\rho\right] =$$

$$= \frac{1}{\pi}\int_0^{1/\beta^2}\exp(-\rho\tau)\sin\beta\xi\sqrt{\rho\left(\frac{1}{\beta^2}-\rho\right)}d\rho.$$

Integration with respect to τ gives

$$\Psi_1(\xi,\tau) = -\frac{1}{\pi}\int_0^{1/\beta^2}\exp(-\rho\tau)\frac{\sin\beta\xi\sqrt{\rho\left(\frac{1}{\beta^2}-\rho\right)}}{\rho}d\rho + C.$$

Under the conditions of the problem, $\Psi_1(0,\tau)=1$; therefore, $C = 1$, and we finally obtain

$$\Psi_1(\xi,\tau) =$$

$$= 1-\frac{1}{\pi}\int_0^{1/\beta^2}\exp(-\rho\tau)\frac{\sin\beta\xi\sqrt{\rho\left(\frac{1}{\beta^2}-\rho\right)}}{\rho}d\rho =$$

$$= \Psi_2(\xi,\tau).$$

Thus, it was shown that

$$W(\xi,\tau) = \Psi_1(\xi,\tau)\eta(\tau-\xi\beta) = \Psi_2(\xi,\tau)\eta(\tau-\xi\beta). \quad (29)$$

Figure 2 presents the curves $W(\xi, \tau)$ in the section $\xi = 2$; both curves calculated from formulas (29) virtually coincide. The above reasoning can also be extended to the Neumann and Cauchy boundary value problems under generalized boundary conditions, which emphasizes the specificity of hyperbolic transfer models.

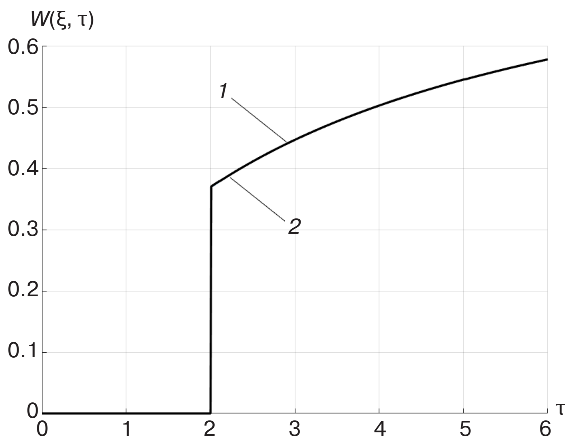

Fig. 2. Results of calculating the function $W(\xi, \tau)$ in the section $\xi = 2$ ($\beta = 1$): point 1—by (26), point 2—by (28)

Canonical finite domains

Mathematical models for Eq. (10) in the domain $\xi \in [0, \xi_0]$, $\tau \geq 0$ under generalized boundary conditions hardly constitute a necessary apparatus of operational calculus, which significantly complicates the determination of their exact analytical solutions. Let us consider a number of operational relations that are characteristic of this case. Find the inverse of the transform

$$\frac{(H-\beta p)^n}{(H+\beta p)^{n+1}}. \quad (30)$$

Use the reference formula [27]

$$\frac{(1-p)^n}{p^{n+1}} = \sum_{k=0}^{n}(-1)^k C_n^k p^{k-n-1} =$$

$$= (-1)^n\sum_{m=0}^{n}(-1)^m C_n^m p^{-(m+1)} \leftarrow$$

$$\leftarrow (-1)^n\sum_{m=0}^{n}(-1)^m C_n^m \frac{1}{m!}t^m =$$

$$= (-1)^n L_n(t),$$

where $L_n(t) = \frac{1}{n!}\exp(t)\frac{d^n}{dt^n}\left[t^n\exp(-t)\right]$ is the Laguerre polynomial [27].

Sequentially using the operational calculus theorems $(1/k)\bar{f}(pk) \leftarrow f(kt)$, $\bar{f}(p-k) \leftarrow \exp(kt)f(t)$, and $\bar{f}(p)\exp(-pt_0) \leftarrow f(t-t_0)\eta(t-t_0)$, we find the sought inverse transform

$$\frac{(H-\beta p)^n}{(H+\beta p)^{n+1}} \leftarrow \frac{(-1)^n}{\beta}\exp\left(-\frac{H}{\beta}t\right)L_n\left(2\frac{H}{\beta}t\right). \quad (31)$$

By similar reasoning, one can show that

$$\frac{(H+\beta p)^n}{(H-\beta p)^{n+1}} \leftarrow \frac{1}{\beta}\exp\left(\frac{H}{\beta}t\right)L_n^*\left(2\frac{H}{\beta}t\right), \quad (32)$$

where $L_n^*(t) = (-1)^{n+1}\sum_{m=0}^{n}C_n^m\frac{1}{m!}t^m$ is the Kartashov polynomial.

Let us find the inverse of the transform $\bar{f}(\sqrt{(p+a)^2-b^2}$, if $\bar{f}(p) \leftarrow f(t)$. Use the Efros theorem

$$\bar{f}\left[\bar{\varphi}_1(p)\right]\bar{\varphi}_2(p) \leftarrow \int_0^t f(\tau)\Psi(\tau,t)d\tau,$$

$$\Psi(\tau,t) \to \exp\left[-\tau\bar{\varphi}_1(p)\right]\bar{\varphi}_2(p). \quad (33)$$

We find

$$\overline{f}(\sqrt{(p+a)^2-b^2}) \leftarrow$$
$$\leftarrow \exp(-at)\left[f(t)+b\int_0^t yf(y)\frac{I_1(b\sqrt{t^2-y^2})}{\sqrt{t^2-y^2}}dy\right]. \quad (34)$$

Similarly, we find

$$\frac{1}{p}\overline{f}(\sqrt{(p+a)^2-b^2})\exp\left[-\gamma\sqrt{(p+a)^2-b^2}\right] \leftarrow$$
$$\leftarrow \int_\gamma^t f(\tau-\gamma)\left[\exp(-a\tau)+b\tau\int_\tau^t \exp(-ay)\frac{I_1(b\sqrt{y^2-\tau^2})}{\sqrt{y^2-\tau^2}}dy\right]d\tau. \quad (35)$$

Relation (35) has a number of special cases, including the practically important expression

$$\left.\begin{aligned}
&\frac{1}{p}\overline{f}(\sqrt{\beta^2p^2+p})\exp(-\gamma\sqrt{\beta^2p^2+p}) \leftarrow\\
&\leftarrow \int_0^t \exp\left(-\frac{\tau}{2\beta^2}\right)\varphi(\gamma,\tau)d\tau,\\
&\varphi(\gamma,t)=f^*(t-\gamma\beta)+\\
&+\frac{1}{2\beta^2}\int_0^t yf^*(y-\gamma\beta)\frac{I_1\left(\frac{1}{2\beta^2}\sqrt{t^2-y^2}\right)}{\sqrt{t^2-y^2}}dy,\\
&f^*(t)=\frac{1}{\beta}f(t/\beta)\eta(t).
\end{aligned}\right\}. \quad (36)$$

Let then $\overline{S}(p)=\sqrt{\beta^2p^2+p}$. From the previous relations, one can find

$$\left.\begin{aligned}
&\frac{\left[H-\overline{S}(p)\right]^n}{\left[H+\overline{S}(p)\right]^{n+1}}\exp\left[-\gamma\overline{S}(p)\right] \leftarrow\\
&\leftarrow \exp\left(-\frac{t}{2\beta^2}\right)[f(t-\gamma\beta]+\\
&+\frac{1}{2\beta^2}\int_0^t f(\tau-\gamma\beta)\frac{\tau I_1\left(\frac{1}{2\beta^2}\sqrt{t^2-\tau^2}\right)}{\sqrt{t^2-\tau^2}}d\tau,\\
&f(t)=\frac{(-1)^n}{\beta}\exp\left(-\frac{H}{\beta}t\right)L_n\left(\frac{2H}{\beta}t\right)\eta(t)
\end{aligned}\right\}. \quad (37)$$

As applications of the obtained results, let us consider two model that are characteristic of thermal shock:

$$\frac{\partial W_i}{\partial\tau}=\frac{\partial^2 W_i}{\partial\xi^2}-\beta^2\frac{\partial^2 W_i}{\partial\tau^2},\ 0<\xi<\xi_0,\ \tau>0,\ (i=1,2), \quad (38)$$

$$W_i|_{\tau=0}=(\partial W_i/\partial\tau)_{\tau=0}=0,\ 0\le\xi\le\xi_0,$$

$$W_1|_{\xi=0}=1,\ W_1|_{\xi=\xi_0}=0,\ \tau>0, \quad (39)$$

$$W_2|_{\xi=0}=1,\ (\partial W_2/\partial\xi)_{\xi=\xi_0}=0,\ \tau>0. \quad (40)$$

In the Laplace transform space, we find

$$\overline{W}_1(\xi,p)=\frac{1}{p}\frac{\sinh\left[(\xi_0-\xi)\overline{S}(p)\right]}{\sinh\left[\xi_0\overline{S}(p)\right]},$$

$$\overline{W}_2(\xi,p)=\frac{1}{p}\frac{\cosh\left[(\xi-\xi_0)\overline{S}(p)\right]}{\cosh\left[\xi_0\overline{S}(p)\right]}.$$

Transform the hyperbolic functions in the fractions:

$$\overline{W}_1(\xi,p)=\sum_{k=0}^{\infty}\frac{1}{p}\left\{\exp\left[-\gamma_{1k}\overline{S}(p)\right]-\exp\left[-\gamma_{2k}\overline{S}(p)\right]\right\},$$

$$\overline{W}_2(\xi,p)=$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{p}\left\{\exp\left[-\gamma_{1k}\overline{S}(p)\right]+\exp\left[-\gamma_{2k}\overline{S}(p)\right]\right\},$$

where

$$\gamma_{1k}=2k\xi_o+\xi,\ \gamma_{2k}=$$
$$=2(k+1)\xi_0+\xi,\ \overline{S}(p)=\sqrt{\beta^2p^2+p}.$$

Calculation of the inverse transforms using the above relations gives

$$W_1(\xi,\tau)=\sum_{k=0}^{\infty}\left\{\Psi\left[\gamma_{1k}(\xi),\tau\right]-\Psi\left[\gamma_{2k}(\xi),\tau\right]\right\}, \quad (41)$$

$$W_2(\xi,\tau)=\sum_{k=0}^{\infty}(-1)^k\left\{\Psi\left[\gamma_{1k}(\xi),\tau\right]+\Psi\left[\gamma_{2k}(\xi),\tau\right]\right\}, \quad (42)$$

where

$$\Psi(\gamma_{ik},\tau)=$$
$$=\left[\exp\left(-\frac{\gamma_{ik}}{2\beta}\right)+\frac{\gamma_{ik}}{2\beta}\int_{\gamma_{ik}/2\beta}^{\tau/2\beta^2}\exp(-y)\frac{I_1\left(\sqrt{y^2-\left(\gamma_{ik}/2\beta\right)^2}\right)}{\sqrt{y^2-\left(\gamma_{ik}/2\beta\right)^2}}dy\right]\times$$
$$\times\,\eta(\tau-\gamma_{ik}\beta). \quad (43)$$

Note that solutions (41)–(43) were previously unknown in analytical thermal physics. All the nine boundary conditions for $W(\xi,\tau)$ in the domain $\xi\in[0,\xi_0]$, $\tau\ge 0$ can be analyzed similarly, and the studied problem for a finite domain can thus be considered resolved. At the time, here, as above, analytical solutions can be obtained as other functional expressions, equivalent to the presented ones. This is one of the specific features of hyperbolic transfer models. Numerical implementation of the obtained relations seems to involve no fundamental difficulties in view of the potential of the existing analytical thermal physics software.

CONCLUSIONS

The inverse transforms of nonstandard operational (Laplace) transforms are presented. These inverse transforms are encountered in operational solutions of a wide class of problems of local nonequilibrium processes of (heat, mass, momentum) transfer, electrical circuits, hydrodynamics, vibrations, thermomechanics, etc. Illustrative examples were given, and it was demonstrated that it is possible to construct analytical solutions of boundary value problems of unsteady-state heat conduction in a partially bounded domain in the form of various functional expressions, for which the equivalence was proven. The presented analytical solutions in canonical domains are new in analytical thermal physics.

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