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RESEARCH ARTICLE

# Evolution of the rotational motion of a viscoelastic planet with a core on an elliptical orbit

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**Abstract.** The work is devoted to the study of the evolution of the rotational motion of the planet in the central Newtonian field of forces. The planet is modeled by a body, consisting of a solid core and a viscoelastic shell rigidly attached to it. A limited formulation of the problem is considered, when the center of mass of the planet moves along a given Keplerian elliptical orbit. The equations of motion are derived in the form of a system of Routh equations using the canonical Andoyer variables, which in the unperturbed problem are "action-angle" variables, and have the form of integro-differential equations with partial derivatives. The technique developed by V.G. Vil'ke is used for mechanical systems with an infinite number of degrees of freedom. A system of ordinary differential equations is obtained by the method of separation of motions, which describes the rotational motion of the planet, taking into account the perturbations caused by elasticity and dissipation. An evolutionary system of equations for the "action" variables and slow angular variables is obtained by the averaging method. A phase portrait is constructed that describes the mutual change in the modulus of the angular momentum vector **G** of the rotational motion and the cosine of the angle between this vector and the normal to the orbital plane of the planet's center of mass. A stationary solution of the evolutionary system of equations is found, which is asymptotically stable. It is shown that in stationary motion the angular momentum vector **G** is orthogonal to the orbital plane, and the limiting value of the modulus of this vector depends on the eccentricity of the elliptical orbit. The constructed mathematical model can be used to study the tidal evolution of the rotational motion of planets and satellites. The results obtained in this work are consistent with the results of previous studies in this area.

**Keywords:** viscoelastic body, Keplerian elliptic orbit, Andoyer variables, averaging method, dissipative evolution of motion

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НАУЧНАЯ СТАТЬЯ

# Эволюция вращательного движения вязкоупругой планеты с ядром на эллиптической орбите

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**Резюме.** Работа посвящена исследованию эволюции вращательного движения планеты в центральном ньютоновском поле сил. Планета моделируется телом, состоящим из твердого ядра и жестко прикрепленной к нему вязкоупругой оболочки. Рассматривается ограниченная постановка задачи, когда центр масс планеты движется по заданной кеплеровской эллиптической орбите. Уравнения движения выводятся в форме системы уравнений Рауса с использованием канонических переменных Андуайе, которые в невозмущенной задаче являются переменными «действие-угол» и имеют вид интегро-дифференциальных уравнений с частными производными. Используется методика, разработанная Вильке В.Г. для механических систем с бесконечным числом степеней свободы. Методом разделения движений получена система обыкновенных дифференциальных уравнений, описывающая вращательное движение планеты с учетом возмущений, вызванных упругостью и диссипацией. Методом усреднения получена эволюционная система уравнений относительно переменных «действие» и медленных угловых переменных. Построен фазовый портрет, описывающий взаимное изменение модуля вектора кинетического момента **G** вращательного движения и косинуса угла между этим вектором и нормалью к плоскости орбиты центра масс планеты. Найдено стационарное решение эволюционной системы уравнений, которое является асимптотически устойчивым. Показано, что в стационарном движении вектор кинетического момента **G** ортогонален плоскости орбиты, а предельное значение модуля этого вектора зависит от эксцентриситета эллиптической орбиты. Построенная математическая модель может быть использована для изучения приливной эволюции вращательного движения планет и спутников. Полученные в работе результаты согласуются с результатами ранее проведенных исследований в этой области.

**Ключевые слова:** вязкоупругое тело, кеплеровская эллиптическая орбита, переменные Андуайе, метод усреднения, диссипативная эволюция движения, метод усреднения

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## INTRODUCTION

The motion of a spherically symmetric rigid body relative to the center of mass, which moves in a Keplerian orbit, is a uniform rotation about an axis oriented constantly in an inertial coordinate system. Because none of the bodies of the Solar System is a rigid body, the interaction with the central body, around which a planet revolves, forms bulges on the viscoelastic body of the planet. These bulges tend to align with the planet–central body line. Because of the internal viscous friction, the tidal bulges lag behind and are shifted at a certain angle to the above line. This gives rise to gravitational torque. In addition, the planet is contracted along the axis of rotation. All of this affects the rate of rotation of the planet.

To describe the dynamics of a system, classical tidal theory typically uses the models of rigid body and point particle. The theory is based on various assumptions of the values of tidal bulges and lag angle [1].

To study the tidal evolution of the rotational motion of celestial bodies, V.V. Beletskii proposed a phenomenological model formula for tidal torque [2, 3] based on the model of rigid body for a planet. Later, this formula was confirmed in V.G. Vil'ke's theory of a viscoelastic sphere in a gravitational field [4].

An evolutionary system of equations of the rotational motion of a viscoelastic sphere in a central Newtonian force field in a circular orbit was obtained [5] by the method of separation of motions and averaging [4].

Previously [6], an evolutionary system of equations was derived for the translational–rotational motion of a viscoelastic sphere in the spatial case in the Andoyer–Delaunay variables. For the planets of the Solar System, the rate of evolution of the angular velocity of the proper rotation of a planet is $10^7$–$10^9$ times higher than the rate of evolution of the mean motion in the orbit because their ratio is equal to the ratio of the squares of the radii of the orbit and the planet [4]. In this work, a restricted formulation of the problem was considered, which enabled one to make a more detailed study of the dissipative evolution of the rotational motion of a planet.

In the 1980s–1990s, the rotational motion of a solid body with elastic and dissipative elements was investigated in many works [7–9]. More recently, the tidal evolution of the rotational motion of celestial bodies have been studied using various models of viscoelastic bodies [10, 11].

## 1. FORMULATION OF THE PROBLEM. EQUATIONS OF MOTION

Let us consider a problem of the motion of a planet relative to the center of mass in a central Newtonian gravitational field. The planet is modeled by a body comprising a rigid core and a viscoelastic shell attached rigidly to the core. In the absence of deformations, i.e., in the natural, undeformed state, the planet occupies region $V$ in the three-dimensional Euclidean space:

$$V = V_0 \cup V_1,\ \ V_0 = \left\{ \mathbf{r} \in E^3 : |\mathbf{r}| \le r_0 \right\},$$
$$V_1 = \left\{ \mathbf{r} \in E^3 : r_0 < |\mathbf{r}| \le r_1 \right\},$$

where $r_0$ and $r_1$ are the inner and outer radii of the shell, respectively. Let $\rho_0$ and $\rho_1$ be the densities of the core and the viscoelastic shell, respectively, which are assumed to be constant; and $m_0$ and $m_1$ are their respective masses.

Let the center of mass of the planet move in a given elliptical orbit. We introduce inertial coordinate system $OXYZ$ with the origin at the attracting center coinciding with one of the foci of the ellipse. Let the $OX$ axis be directed along the radius vector of the perigee; the $OZ$ axis, perpendicular to the plane of the orbit, and the $OY$ axis, so that the unit vectors of the fixed coordinate systems form a right-hand system. To describe the rotational motion of the planet, we introduce moving coordinate system $Cx_1x_2x_3$ and König system of axes $C\xi_1\xi_2\xi_3$ with the origin at the center of mass $C$ of the planet.

The position of point $M$ on the planet in the inertial coordinate system $OXYZ$ is determined by the vector field

$$\mathbf{R}_M(\mathbf{r},t) = \mathbf{R}(t) + \Gamma(t)\big(\mathbf{r} + \mathbf{u}(\mathbf{r},t)\big), \qquad (1.1)$$

$$\mathbf{R}(t) = \frac{1}{m}\int_V \mathbf{R}_M(\mathbf{r},t)\rho dx,\quad \int_{V_1} \mathbf{u} dx = 0,\quad \int_{V_1} \operatorname{rot}\mathbf{u} dx = 0, \quad (1.2)$$

where $\mathbf{R}(t)$ is the radius vector of the center of mass of the planet; $\Gamma = \Gamma(t)$ is the operator of transition from the moving coordinate system $Cx_1x_2x_3$ to the König system of axes $C\xi_1\xi_2\xi_3$; $\mathbf{u}(\mathbf{r},t)$ is the elastic displacement vector, which is identically zero for points of the rigid core $V_0$; $m = m_0 + m_1$; and $\rho = \rho_i$ for $\mathbf{r} \in V_i$, $(i = 0, 1)$. Conditions (1.2) uniquely determine the radius vector of the center of mass $C$ of the deformed planet, and also the moving coordinate system $Cx_1x_2x_3$, relative to which the viscoelastic planet does not rotate in the integral sense [4]. In the coordinate system $Cx_1x_2x_3$,

$$\mathbf{u}(\mathbf{r},t) = \big(u_1(\mathbf{r},t),\, u_2(\mathbf{r},t), u_3(\mathbf{r},t)\big),\ \mathbf{r} = \big(x_1, x_2, x_3\big).$$

The problem is solved within a linear model of elasticity theory. The functional of the potential energy of elastic deformations has the form

$$\mathcal{E} = \int_{V_1} \mathcal{E}[\mathbf{u}] dx,\ \ \mathcal{E}[\mathbf{u}] = \alpha_1\left(I_E^2 - \alpha_2 II_E\right), \qquad (1.3)$$

$$\alpha_1 = \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)},\ \ \alpha_2 = \frac{2(1-2\nu)}{1-\nu},$$
$$\alpha_1 > 0,\ \ 0 < \alpha_2 < 3,$$

$$I_E = \sum_{j=1}^{3} e_{jj},\ \ II_E = \sum_{k<l}\left(e_{kk}e_{ll} - e_{kl}^2\right),$$
$$e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right),\ \ dx = dx_1 dx_2 dx_3,$$

where $E$ is Young's modulus, and $\nu$ is Poisson's ratio of the viscoelastic shell of the planet.

To describe the dissipative properties of the shall of the planet, we use the Kelvin–Voigt model, i.e., assume that the dissipative functional $\mathcal{D}$ is related to functional (1.3) by the expressions

$$\mathfrak{D} = \int_{V_1} \mathfrak{D}[\dot{\mathbf{u}}]dx, \ \mathfrak{D}[\dot{\mathbf{u}}] = \chi\mathcal{E}[\dot{\mathbf{u}}],$$

where $\chi > 0$ is the coefficient of internal viscous friction.

According to the considered restricted formulation of the problem, the center of mass of the problem moves in a Keplerian elliptical orbit, i.e., the radius vector $\mathbf{R}(t)$ of the point $C$ is a given function of time according to the relations

$$\mathbf{R} = R(\cos\vartheta; \sin\vartheta; 0), \tag{1.4}$$

$$R = \frac{a(1-e^2)}{1+e\cos\vartheta}, \ \dot{\vartheta} = \frac{\partial\vartheta}{\partial l}\dot{l} = \frac{(1+e\cos\vartheta)^2}{(1-e^2)^{3/2}}n,$$

$$n = \sqrt{\frac{\gamma}{a^3}}, \ l = n(t-t_0). \tag{1.5}$$

Here, $\vartheta$ is the true anomaly; $a$ is the semi-major axis of the orbit; $e$ is the eccentricity; $n$ is the mean motion of the center of mass $C$ of the planet in the orbit; $l$ is the mean anomaly; $\gamma$ is the standard gravitational parameter ($\gamma = fM_0$, where $f$ is the universal gravitational constant, and $M_0$ is the mass of the attracting center); and $t_0$ and $t$ are the initial and current times, respectively.

The kinetic energy of the sphere is represented by the functional

$$T = \frac{1}{2}\int_V \dot{\mathbf{R}}_M^2 \rho dx = \frac{1}{2}\int_V \left[\Gamma^{-1}\dot{\mathbf{R}} + \boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u}) + \dot{\mathbf{u}}\right]^2 \rho dx, \tag{1.6}$$

where $\boldsymbol{\omega}\times(\cdot) = \Gamma^{-1}\dot{\Gamma}(\cdot)$, $\boldsymbol{\omega}$ is the angular velocity of the rotation of the sphere (the coordinate system $Cx_1x_2x_3$). Under conditions (1.2), the functional of the kinetic energy of the viscoelastic sphere takes the form

$$T = \frac{1}{2}m\dot{\mathbf{R}}^2 + \frac{1}{2}\int_V [\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u})]^2 \rho dx + \int_{V_1} \big(\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u}), \dot{\mathbf{u}}\big)\rho_1 dx + \frac{1}{2}\int_{V_1} \dot{\mathbf{u}}^2 \rho_1 dx. \tag{1.7}$$

The potential energy of the gravitational field has the form

$$\Pi = -\gamma\int_V \frac{\rho dx}{\sqrt{(\mathbf{R}+\Gamma(\mathbf{r}+\mathbf{u}))^2}}. \tag{1.8}$$

Because $|\mathbf{R}| \gg |\mathbf{r}+\mathbf{u}|$, the integrand in expression (1.8) can be expanded into a series in powers of $|\mathbf{r}+\mathbf{u}|/R$. Truncating the series after the terms of the second order in $|\mathbf{r}+\mathbf{u}|/R$ and of the first order in $|\mathbf{u}|/R$, we obtain

$$\Pi = -\frac{\gamma m}{R} + \frac{\gamma}{R^3}\int_{V_1} [(\mathbf{r},\mathbf{u}) - 3(\xi,\mathbf{r})(\xi,\mathbf{u})]\rho_1 dx,$$

$$\xi = \Gamma^{-1}\mathbf{R}/R. \tag{1.9}$$

The configurational space of the mechanical system is the direct product $SO(3)\times\mathfrak{B}$, where

$$\mathfrak{B} = \left\{\mathbf{u}:\mathbf{u}\in\left(W_2^1(V_1)\right)^3, \int_{V_1}\mathbf{u}dx = 0, \int_{V_1}\operatorname{rot}\mathbf{u}dx = 0, \mathbf{u}\big|_{|\mathbf{r}|=r_0} = 0\right\},$$

$\left(W_2^1(V_1)\right)^3$ is the Sobolev space [4], and $SO(3)$ is the group of rotations of the three-dimensional Euclidean space. The generalized coordinates $q_1q_2q_3$, which determine the group of rotations $SO(3)$, can be, e.g., the Euler angles.

The components of the angular velocity vector $\boldsymbol{\omega}$ are linear homogeneous functions of the generalized velocities $\dot{q}_i$ $(i = 1,2,3)$. Grouping the second-, first-, and zero-degree term of the right-hand side of expression (1.7) that contain the generalized velocities $\dot{q}_i$ $(i = 1,2,3)$, one can represent the kinetic energy functional in the form

$$T = T_2 + T_1 + T_0, \tag{1.10}$$

$$T_2 = \frac{1}{2}\int_V [\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u})]^2 \rho dx, \quad T_1 = \int_{V_1} \big(\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u}), \dot{\mathbf{u}}\big)\rho_1 dx,$$

$$T_0 = \frac{1}{2}m\dot{\mathbf{R}}^2 + \frac{1}{2}\int_{V_1} \dot{\mathbf{u}}^2 \rho_1 dx. \tag{1.11}$$

Let us obtain the equations of motion of the planet in the form of the Routh equations using the canonical Andoyer variables $(\mathbf{I}, \boldsymbol{\varphi}) = (I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3)$ [4, 12] to describe the rotational motion of the coordinate system $Cx_1x_2x_3$ relative to the König axes and the Lagrangian coordinates $u_i(\mathbf{r}, t)$, $i = (1, 2, 3)$ to characterize the deformations of the viscoelastic shell of the planet.

The vector of the angular momentum of the planet about the center of mass is

$$\mathbf{G} = \nabla_\omega T = J[\mathbf{u}]\boldsymbol{\omega} + \mathbf{G}_\mathbf{u}, \tag{1.12}$$

$$J[\mathbf{u}]\boldsymbol{\omega} = \int_V (\mathbf{r}+\mathbf{u})\times[\boldsymbol{\omega}\times(\mathbf{r}+\mathbf{u})]\rho dx,$$

$$\mathbf{G}_\mathbf{u} = \int_{V_1} [(\mathbf{r}+\mathbf{u})\times\dot{\mathbf{u}}]\rho_1 dx. \tag{1.13}$$

Using expression (1.13), the term $T_2$ of the right-hand side of formula (1.10) is expressed in terms of the inertia tensor $J[\mathbf{u}]$ of the deformed planet as

$$T_2 = \frac{1}{2}\big(J[\mathbf{u}]\boldsymbol{\omega}, \boldsymbol{\omega}\big). \tag{1.14}$$

Let us construct the kinetic momentum vector **G** at the point $C$ and also construct the plane $CMN$, which is perpendicular to the vector **G** and intersects the plane $C\xi_1\xi_2$ at the straight line $CM$ and the plane $Cx_1x_2$ at the straight line $CN$. The variable $I_2$ is the magnitude of the vector **G**; and $I_1$ and $I_3$ are its projections on the $Cx_3$ and $C\xi_3$ axes, respectively. The transition from the König system of axes $C\xi_1\xi_2\xi_3$ to the moving coordinate system $Cx_1x_2x_3$ in the Andoyer variables is performed by five successive rotations by angles $\varphi_3$, $\delta_1$, $\varphi_2$, $\delta_2$, and $\varphi_1$ about the $C\xi_3$ axis, the $CM$ axis, the vector **G**, the $CN$ axis, and the $Cx_3$ axis, respectively (Fig. 1).

The transition operator $\Gamma$ in the Andoyer variables is represented as the product of five orthogonal matrices [12]:

$$\Gamma = \Gamma_3(\varphi_3)\Gamma_1(\delta_1)\Gamma_3(\varphi_2)\Gamma_3(\delta_2)\Gamma_3(\varphi_1),$$
$$\cos\delta_1 = I_3/I_2, \ \cos\delta_2 = I_1/I_2,$$

$$\Gamma_3(\varphi_k) = \begin{pmatrix} \cos\varphi_k & -\sin\varphi_k & 0 \\ \sin\varphi_k & \cos\varphi_k & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma_1(\delta_j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_j & -\sin\delta_j \\ 0 & \sin\delta_j & \cos\delta_j \end{pmatrix}.$$

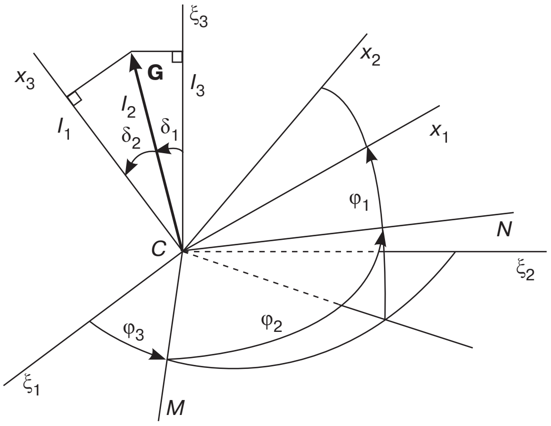


**Fig. 1.** Andoyer variables

In the coordinate system $Cx_1x_2x_3$,

$$\mathbf{G} = \left(\sqrt{I_2^2 - I_1^2}\sin\varphi_1, \sqrt{I_2^2 - I_1^2}\cos\varphi_1, I_1\right), \quad (1.15)$$

$$\xi = \Gamma^{-1}\mathbf{R}/R = \Gamma_3(-\varphi_1)\Gamma_1(-\delta_2)\Gamma_3(-\varphi_2)\times$$
$$\times\Gamma_1(-\delta_1)\Gamma_3(-\varphi_3)(\cos\vartheta, \sin\vartheta, 0)^{\mathrm{T}}. \quad (1.16)$$

It follows from expression (1.12) that

$$\boldsymbol{\omega} = J^{-1}[\mathbf{u}](\mathbf{G} - \mathbf{G_u}). \quad (1.17)$$

Then, from expressions (1.14) and (1.17), the functional $T_2$ can be represented as

$$T_2 = \frac{1}{2}\left(\mathbf{G} - \mathbf{G_u}, J^{-1}[\mathbf{u}](\mathbf{G} - \mathbf{G_u})\right). \quad (1.18)$$

The Routh functional $\mathfrak{R}$, which depends on the canonical variables **I**, $\boldsymbol{\varphi}$ and the Lagrangian variables $\dot{\mathbf{u}}, \mathbf{u}$, is defined by the expression

$$\mathfrak{R} = T_2 - T_0 + \Pi + \mathcal{E}[\mathbf{u}], \quad (1.19)$$

where $T_2$ is given by formula (1.18); and in expression (1.9) for the functional $\Pi$ of the potential energy of the gravitational field, the vector $\xi$ is given by formula (1.16).

Using expressions (1.11) and (1.18), Routh functional (1.19) can be written as

$$\mathfrak{R} = \frac{I_2^2}{2A} - \frac{1}{A}\left(\mathbf{G}, \int\limits_{V_1} \mathbf{r}\times\dot{\mathbf{u}}\rho_1 dx\right) - \frac{1}{2A^2}\times$$
$$\times(J_1[\mathbf{u}]\mathbf{G}, \mathbf{G}) - \frac{1}{2}m\dot{\mathbf{R}}^2 + \Pi + \mathcal{E}[\mathbf{u}] + \mathfrak{R}^*, \quad (1.20)$$

where A is the moment of inertia of the undeformed planet about the diameter, and $\mathfrak{R}^*$ contains terms of the second and higher orders in the coordinates of the vectors **u** and $\dot{\mathbf{u}}$:

$$J_1[\mathbf{u}]\boldsymbol{\omega} = \int\limits_{V_1}\left(\mathbf{r}\times[\boldsymbol{\omega}\times\mathbf{u}] + \mathbf{u}\times[\boldsymbol{\omega}\times\mathbf{r}]\right)\rho_1 dx,$$

$$A = \frac{8\pi}{15}\left[\rho_0 r_0^5 + \rho_1\left(r_1^5 - r_0^5\right)\right].$$

The equations of the rotational motion of the planet in the elliptical orbit are written in the form of the canonical equations in the Andoyer variables and in the form of the d'Alembert–Lagrange variational principle [4]:

$$\dot{I}_k = -\frac{\partial\mathfrak{R}}{\partial\varphi_k}, \quad \dot{\varphi}_k = \frac{\partial\mathfrak{R}}{\partial I_k}, \quad k = 1, 2, 3, \quad (1.21)$$

$$\left(-\frac{d}{dt}\nabla_{\dot{\mathbf{u}}}\mathfrak{R} + \nabla_{\mathbf{u}}\mathfrak{R} + \nabla_{\dot{\mathbf{u}}}\mathfrak{D} + \boldsymbol{\lambda}_1, \delta\mathbf{u}\right)_{V_1} +$$
$$+\int\limits_{V_1}\left(\boldsymbol{\lambda}_2, \operatorname{rot}\delta\mathbf{u}\right)dx = 0, \ \forall\delta\mathbf{u}\in(W_2^1(V))^3. \quad (1.22)$$

Here, $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are Lagrange's undetermined multipliers generated by conditions (1.2).

## 2. DEFORMATION OF THE VISCOELASTIC SHELL OF THE PLANET

Let the stiffness of the deformable shell of the planet be high; i.e., the dimensionless parameter $\tilde{\varepsilon} = \omega^2(0)\rho_1 r_1^2 E^{-1}$ be small (where $\omega(0)$ is the magnitude of the initial angular velocity of the planet). Choosing the scales of the dimensional quantities in a certain manner, one can introduce small parameter $\varepsilon = E^{-1}$. As an unperturbed problem, we consider a problem of the motion of a spherically symmetric rigid-body planet in an elliptical orbit. In this case, $\mathbf{u}(\mathbf{r}, t) = 0$, and the parameter ε is assumed to be zero. The equations of the unperturbed motion have the form.

$$\dot{I}_k = 0,\ k = 1,2,3,\ \dot{\varphi}_1 = 0,\ \dot{\varphi}_2 = I_2/A,\ \dot{\varphi}_3 = 0. \quad (2.1)$$

Equations (2.1) describe the uniform rotation of the planet about one of the diameters at the angular velocity $\dot{\varphi}_2 = I_2/A$. At $\varepsilon \neq 0$, according to the method of separation of motions [4], after the damping of the natural vibrations of the viscoelastic sphere, the solution $\mathbf{u}(\mathbf{r}, t)$ is sought as a series of powers of the small parameter ε:

$$\mathbf{u}(\mathbf{r},t) = \varepsilon\mathbf{u}_1(\mathbf{r},t) + \varepsilon^2\mathbf{u}_2(\mathbf{r},t) + \cdots \quad (2.2)$$

Lagrange's undetermined multipliers $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ should also be sought as series of powers of ε:

$$\lambda_1(t) = \lambda_{10}(t) + \varepsilon\lambda_{11}(t) + \cdots,$$
$$\lambda_2(t) = \lambda_{20}(t) + \varepsilon\lambda_{21}(t) + \cdots \quad (2.3)$$

Using expressions (1.20) and (1.9), Eq. (1.22) for the first-approximation function $\mathbf{u}_1(\mathbf{r}, t)$ is transformed to the form

$$\int_{V_1}\left[-\frac{\rho_1}{A}\frac{d}{dt}(\mathbf{G}\times\mathbf{r}) + \frac{1}{2A^2}(\nabla_{\mathbf{u}}J_1[\mathbf{u}]\mathbf{G},\mathbf{G}) - \frac{\gamma\rho_1\mathbf{r}}{R^3} + \frac{3\gamma\rho_1}{R^3}(\xi,\mathbf{r})\xi + \boldsymbol{\lambda}_{10}\right]\delta\mathbf{u}dx + \int_{\partial V_1}(\boldsymbol{\lambda}_{20}\times\mathbf{n})\delta\mathbf{u}d\sigma - \varepsilon(\nabla_{\mathbf{u}}\mathcal{E}[\mathbf{u}_1 + \chi\dot{\mathbf{u}}_1],\delta\mathbf{u})_{V_1} = 0. \quad (2.4)$$

The derivation of this formula included the use of the divergence theorem in the form:

$$\int_{V_1}\boldsymbol{\lambda}_{20}\operatorname{rot}\delta\mathbf{u}dx = \int_{\partial V_1}(\delta\mathbf{u}\times\boldsymbol{\lambda}_{20})\mathbf{n}d\sigma,$$

where $\partial V_1$ is the boundary of the region $V_1$, and $\mathbf{n}$ is a normal to $\partial V_1$.

The variables $(\mathbf{I}, \boldsymbol{\varphi})$ in Eq. (2.4) according to the method of separation of motions are solutions of unperturbed problem (2.1). Therefore, in Eq. (2.4), $\frac{d}{dt}(\mathbf{G}\times\mathbf{r}) = 0.$ Further,

$$(\nabla_{\mathbf{u}}J_1[\mathbf{u}]\mathbf{G},\mathbf{G}) = 2\rho_1 I_2^2\mathbf{r} - 2\rho_1(\mathbf{r},\mathbf{G})\mathbf{G}. \quad (2.5)$$

Successively substituting $\delta\mathbf{u} = \delta\boldsymbol{\alpha}\times\mathbf{r}$ and $\delta\mathbf{u} = \mathbf{a}$, ($\delta\boldsymbol{\alpha}$, $\mathbf{a} \in E^3$), into Eq. (2.4) and taking into account that the work done by the elastic and dissipative forces at infinitesimal rotations is zero, we obtain $\boldsymbol{\lambda}_{10} = 0$ and $\boldsymbol{\lambda}_{20} = 0$.

For the last term of the left-hand side of Eq. (2.4), the following equality is valid [4]:

$$\left(\nabla_{\mathbf{u}}\mathcal{E}[\mathbf{u}],\delta\mathbf{u}\right)_{V_1} = = \int_{V_1}\nabla\mathcal{E}[\mathbf{u}]\delta\mathbf{u}dx + \int_{\partial V_1}\sum_{i=1}^{3}\sigma_{ni}\,\delta u_i dx,$$

$$\nabla\mathcal{E}[\mathbf{u}] = -\frac{E}{2(1+\nu)}\left(\frac{1}{1-2\nu}\operatorname{grad}\operatorname{div}\mathbf{u} + \Delta\mathbf{u}\right),$$

$$\sigma_{ni}[\mathbf{u}] = \frac{E\nu\gamma_i}{(1+\nu)(1-2\nu)}\operatorname{div}\mathbf{u} + \frac{E}{2(1+\nu)}\left(\frac{\partial\mathbf{u}}{\partial x_i} + \operatorname{grad}u_i,\mathbf{n}\right),$$
$$i = 1,2,3,\ \mathbf{n} = (\gamma_1,\gamma_2,\gamma_3).$$

Thus, the boundary-value problem to determine the first-approximation function $\mathbf{u}_1(\mathbf{r}, t)$ takes the form

$$\varepsilon\nabla\mathcal{E}[\mathbf{u}_1 + \chi\dot{\mathbf{u}}_1] = = \rho_1\frac{I_2^2}{A^2}\mathbf{r} - \frac{\rho_1}{A^2}(\mathbf{r},\mathbf{G})\mathbf{G} - \frac{\rho_1\gamma}{R^3}\mathbf{r} + \frac{3\rho_1\gamma}{R^3}(\xi,\mathbf{r})\xi, \quad (2.6)$$

$$\mathbf{u}_1\big|_{|\mathbf{r}|=r_0} = 0,\ \sigma_{ni}[\mathbf{u}_1]\big|_{|\mathbf{r}|=r_1} = 0,\ i = (1,2,3). \quad (2.7)$$

Boundary conditions (2.7) mean that the movements on the inner boundary of the viscoelastic shell of the planet are zero, and so are the stresses on its outer boundary. The solution of boundary-value problem (2.6)–(2.7) has the form [13, 14]

$$\mathbf{u}_1 = \mathbf{u}_{10} + \mathbf{u}_{11} + \mathbf{u}_{12}, \quad (2.8)$$

$$\mathbf{u}_{10} = \frac{2I_2^2}{3A^2}\rho_1\left(a_1 r^2 + a_2 + \frac{a_3}{r^3}\right)\mathbf{r},$$

$$\mathbf{u}_{11} = \rho_1\left\{p(r_0,r_1,\nu)\left[\frac{I_2^2}{3A^2}\mathbf{r} - \frac{\mathbf{G}}{A^2}(\mathbf{G},\mathbf{r})\right] + q(r_0,r_1,\nu)\left[\frac{I_2^2}{6A^2}r^2 - \frac{1}{2A^2}(\mathbf{G},\mathbf{r})^2\right]\mathbf{r}\right\},$$

$$\mathbf{u}_{12} = -\frac{3\gamma\rho_1}{R^3}\left(1+\frac{3\chi\dot{R}}{R}\right)\times$$
$$\times\left\{p(r_0,r_1,\nu)\left[\frac{1}{3}\mathbf{r}-\xi(\xi,\mathbf{r})\right]+\right.$$
$$\left.+q(r_0,r_1,\nu)\left[\frac{1}{6}r^2-\frac{1}{2}(\xi,\mathbf{r})^2\right]\mathbf{r}\right\}-$$
$$-\frac{3\chi\gamma\rho_1}{R^3}\left\{p(r_0,r_1,\nu)\left[\dot{\xi}(\xi,\mathbf{r})+\xi(\dot{\xi},\mathbf{r})\right]+\right.$$
$$\left.+q(r_0,r_1,\nu)(\xi,\mathbf{r})(\dot{\xi},\mathbf{r})\mathbf{r}\right\}, \quad (2.9)$$

$$a_1 = -\frac{1+\nu}{5(k+2)}, \quad a_2 = -\frac{a_1 r_1^2\left(4x^5+5k+6\right)}{4x^3+3k+2},$$

$$a_3 = -\frac{a_1 r_1^5 x^3\left((3k+2)x^2-5k-6\right)}{4x^3+3k+2},$$

$$p(r_0,r_1,\nu) = b_1 r^2 + b_2 + \frac{b_3}{r^3}+\frac{b_4}{r^5},$$
$$q(r_0,r_1,\nu) = b_5 + \frac{b_6}{r^5}+\frac{b_7}{r^7},$$

$$k = \frac{2\nu}{1-2\nu}, \quad r = |\mathbf{r}|, \quad x = \frac{r_0}{r_1},$$

$$b_1 = -\frac{(1+\nu)}{\Delta_0}\left\{8(9k+14)x^{10}+80x^7+24(k+1)\times\right.$$
$$\left.\times(5k+11)x^5-5(k+2)(15k+16)x^3+2(3k+8)(5k+4)\right\},$$

$$b_2 = \frac{(1+\nu)r_1^2}{\Delta_0}\left\{8(9k+14)x^{12}+8\left(15k^2+46k+51\right)x^7-\right.$$
$$\left.-\left(63k^2+114k+56\right)x^5+4(3k+8)(4k+3)\right\},$$

$$b_3 = \frac{2(1+\nu)r_1^5x^3}{\Delta_0}\times$$
$$\times\left\{40x^9-16(k+6)x^7+(21k+16)x^2-10(4k+3)\right\},$$

$$b_4 = \frac{2(1+\nu)(k+1)r_1^7x^5}{\Delta_0}\times$$
$$\times\left\{24x^7-2(3k+26)x^5+(15k+16)x^2-6(4k+3)\right\},$$

$$b_5 = -\frac{4(1+\nu)(k+1)}{\Delta_0}\times$$
$$\times\left\{60x^7-12(2k+17)x^5+5(3k+26)x^3-2(3k+8)\right\},$$

$$b_6 = 3(k+1)b_3, \; b_7 = -5b_4,$$

$$\Delta_0 = 8(2k+7)(9k+14)x^{10}+200\left(3k^2+8k+7\right)x^7-$$
$$-1008(k+1)^2x^5+25\left(27k^2+56k+28\right)x^3+$$
$$+2(3k+8)(19k+14).$$

In expression (2.9), the time differentiation is performed on the strength of unperturbed system of equations of motion (2.1), and the $\mathbf{G}$ and $\xi$ values are found from formulas (1.15) and (1.16).

## 3. PERTURBED SYSTEM OF EQUATIONS OF MOTION

The found solution $\mathbf{u} = \varepsilon\mathbf{u}_1 = \varepsilon(\mathbf{u}_{10} + \mathbf{u}_{11} + \mathbf{u}_{12})$ describes the forced vibrations of the viscoelastic sphere. According to the asymptotic method of separation of motions, this solution should further be substituted into the right-hand sides of Eqs. (1.21) for the "slow" variables after preliminary linearization of them in $\dot{\mathbf{u}}$ and $\mathbf{u}$. This substitution with subsequent calculation of the triple integrals over the spherical layer $V_1$ gives the perturbed system of equations of the rotational motion of the planet:

$$\dot{I}_1 = -\frac{6\varepsilon\gamma\rho_1^2 D}{A^2R^3}\left(1+3\chi\frac{\dot{R}}{R}\right)\left(\frac{\partial\mathbf{G}}{\partial\varphi_1},\xi\right)(\mathbf{G},\xi)+$$
$$+\frac{6\varepsilon\chi\gamma\rho_1^2 D}{A^2R^3}\left\{\left(\frac{\partial\mathbf{G}}{\partial\varphi_1},\xi\right)(\mathbf{G},\dot{\xi})+\left(\frac{\partial\mathbf{G}}{\partial\varphi_1},\dot{\xi}\right)(\mathbf{G},\xi)\right\}-$$
$$-\frac{6\varepsilon\gamma\rho_1^2 D}{A^2R^3}\left(\frac{\partial\xi}{\partial\varphi_1},\mathbf{G}\right)(\xi,\mathbf{G})-\frac{18\varepsilon\chi\gamma^2\rho_1^2 D}{R^6}\left(\frac{\partial\xi}{\partial\varphi_1},\dot{\xi}\right),$$

$$\dot{I}_j = -\frac{6\varepsilon\gamma\rho_1^2 D}{A^2R^3}\left(\frac{\partial\xi}{\partial\varphi_j},\mathbf{G}\right)(\xi,\mathbf{G})-$$
$$-\frac{18\varepsilon\chi\gamma^2\rho_1^2 D}{R^6}\left(\frac{\partial\xi}{\partial\varphi_j},\dot{\xi}\right), \; j=2,3, \quad (3.1)$$

$$\dot{\varphi}_1 = \frac{6\varepsilon\gamma\rho_1^2 D}{A^2R^3}\left(1+3\chi\frac{\dot{R}}{R}\right)\left(\frac{\partial\mathbf{G}}{\partial I_1},\xi\right)(\mathbf{G},\xi)-$$
$$-\frac{6\varepsilon\chi\gamma\rho_1^2 D}{A^2R^3}\left\{\left(\frac{\partial\mathbf{G}}{\partial I_1},\xi\right)(\mathbf{G},\dot{\xi})+\left(\frac{\partial\mathbf{G}}{\partial I_1},\dot{\xi}\right)(\mathbf{G},\xi)\right\}+$$
$$+\frac{6\varepsilon\gamma\rho_1^2 D}{A^2R^3}\left(\frac{\partial\xi}{\partial I_1},\mathbf{G}\right)(\xi,\mathbf{G})+\frac{18\varepsilon\chi\gamma^2\rho_1^2 D}{R^6}\left(\frac{\partial\xi}{\partial I_1},\dot{\xi}\right),$$

$$\dot{\varphi}_3 = \frac{6\varepsilon\gamma\rho_1^2 D}{A^2R^3}\left(\frac{\partial\xi}{\partial I_3},\mathbf{G}\right)(\xi,\mathbf{G})+\frac{18\varepsilon\chi\gamma^2\rho_1^2 D}{R^6}\left(\frac{\partial\xi}{\partial I_3},\dot{\xi}\right).$$

Here, $D = \frac{1}{3}k_1(5b_1 + b_5) + b_2k_3 + \frac{1}{5}k_4(5b_3 + b_6)$, $k_1 = \frac{4\pi}{35}(r_1^7 - r_0^7)$, $k_3 = \frac{4\pi}{15}(r_1^5 - r_0^5)$, and $k_4 = \frac{2\pi}{3}(r_1^2 - r_0^2)$.

The variable $\varphi_2$ is the fast angular variable, and $\dot\varphi_2 \approx I_2/A$. The time differentiation on the right-hand sides of the system of Eqs. (3.2) is performed on the strength of unperturbed system (2.1) and expressions (1.5); i.e.,

$$\dot R = \frac{\partial R}{\partial \vartheta}\dot\vartheta = \frac{ane\sin\vartheta}{\sqrt{1-e^2}},$$

$$\dot\xi = \frac{\partial \xi}{\partial \vartheta}\dot\vartheta + \frac{\partial \xi}{\partial \varphi_2}\dot\varphi_2 = \frac{\partial \xi}{\partial \vartheta}\frac{(1+e\cos\vartheta)^2}{(1-e^2)^{3/2}}n + \frac{\partial \xi}{\partial \varphi_2}\frac{I_2}{A}.$$

Here, the vector $\xi$ is defined by formula (1.16) and can be represented as

$$\xi = (\xi_x, \xi_y, \xi_z),$$

$$\xi_x = d_x\cos\varphi_1 + d_y\sin\varphi_1,$$

$$\xi_y = -d_x\sin\varphi_1 + d_y\cos\varphi_1,\ \xi_z = d_z,$$

$$d_x = \cos\varphi_2\cos(\varphi_3 - \vartheta) - \sin\varphi_2\cos\delta_1\sin(\varphi_3 - \vartheta),$$

$$d_y = -\sin\varphi_2\cos(\varphi_3 - \vartheta)\cos\delta_2 - \\ -\cos\varphi_2\cos\delta_1\sin(\varphi_3 - \vartheta)\cos\delta_2 + \\ +\sin\delta_1\sin(\varphi_3 - \vartheta)\sin\delta_2,$$

$$d_z = \sin\varphi_2\cos(\varphi_3 - \vartheta)\sin\delta_2 + \\ +\cos\varphi_2\cos\delta_1\sin(\varphi_3 - \vartheta)\sin\delta_2 + \\ +\sin\delta_1\sin(\varphi_3 - \vartheta)\cos\delta_2.$$

## 4. EVOLUTIONARY SYSTEM OF EQUATIONS OF MOTION

Next, let us average the right-hand sides of perturbed system (3.1) over the fast angular variables—the Andoyer variable $\varphi_2$ and the mean anomaly $l$—provided that there are no resonances. The averaging procedure is the calculation of the integral

$$\langle * \rangle_{\varphi_2, l} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}(*)d\varphi_2 dl = \\ = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}(*)\frac{(1-e^2)^{3/2}}{(1+e\cos\vartheta)^2}d\varphi_2 d\vartheta.$$

This gives the evolutionary system of equations of the dynamics of the rotational motion of the viscoelastic planet with then rigid core in the form

$$\dot I_1 = \dot I_2\cos\delta_2,$$

$$\dot I_2 = -\frac{18\chi\rho_1^2\varepsilon Dn^4}{(1-e^2)^{9/2}}\left\{\frac{I_2}{A}\left[\frac{1}{2} + \frac{3e^2}{4}(1+2\cos^2\varphi_3) + \frac{e^4}{16}(1+4\cos^2\varphi_3) + \cos^2\delta_1\left(\frac{1}{2} + \frac{3e^2}{4}(1+2\sin^2\varphi_3) + \frac{e^4}{16}(1+4\sin^2\varphi_3)\right)\right] - \frac{n\cos\delta_1}{(1-e^2)^{3/2}}\cdot F_2(e)\right\}, \quad (4.1)$$

$$\dot I_3 = -\frac{18\chi\rho_1^2\varepsilon Dn^4}{(1-e^2)^{9/2}}\left\{\frac{I_2}{A}\cdot\cos\delta_1\cdot F_1(e) - \frac{n}{(1-e^2)^{3/2}}\cdot F_2(e)\right\},$$

$$\dot\varphi_1 = 0,\ \dot\varphi_2 \approx I_2/A,$$

$$\dot\varphi_3 = -\frac{3\rho_1^2\varepsilon DI_2n^2\cos\delta_1}{A^2(1-e^2)^{3/2}} + \frac{9\chi\rho_1^2\varepsilon Dn^4}{A(1-e^2)^{9/2}}\left(\frac{3e^2}{2} + \frac{e^4}{4}\right)\sin 2\varphi_3,$$

where $F_1(e) = 1 + 3e^2 + \frac{3}{8}e^4$, $F_2(e) = 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6$.

It follows from the first equation of system (4.1) that the angle between the kinetic momentum vector **G** and the $Cx_3$ axis is conserved:

$$\cos\delta_2 = \frac{I_1}{I_2} = \frac{I_1(0)}{I_2(0)}.$$

The method of separation of motions is based on the physical assumption that the time of damping of the free natural vibrations of the elastic medium at the lowest frequency is longer than the period of these vibrations, but much shorter than the characteristic time of the rigid-body motion [7, 8]. Therefore, the product $\chi n$ is small ($\chi n \ll 1$). Consequently, in the system of Eqs. (4.1), the angular variable $\varphi_3$ can be considered fast, and the averaging in $\varphi_3$ can be performed. The averaging transforms system (4.1) to the form

$$\dot I_1 = \dot I_2\cos\delta_2,$$
$$\dot I_2 = -\frac{18\chi\rho_1^2\varepsilon Dn^4}{(1-e^2)^{9/2}}\left\{\frac{I_2}{2A}F_1(e)(1+\cos^2\delta_1) - \frac{n\cos\delta_1}{(1-e^2)^{3/2}}\cdot F_2(e)\right\},$$
$$\dot I_3 = -\frac{18\chi\rho_1^2\varepsilon Dn^4}{(1-e^2)^{9/2}}\left\{\frac{I_2}{A}\cdot\cos\delta_1\cdot F_1(e) - \frac{n}{(1-e^2)^{3/2}}\cdot F_2(e)\right\},$$
$$\dot\varphi_1 = 0,\quad \dot\varphi_2 \approx I_2/A,\quad \dot\varphi_3 = -\frac{3\rho_1^2\varepsilon DI_2n^2\cos\delta_1}{A^2(1-e^2)^{3/2}}. \quad (4.2)$$

Let us pass from the variables $I_2$, $I_3$ to the dimensionless variables $y = \cos\delta_1 = I_3/I_2$, $\omega_0 = I_2/An$. Then, from the second and third equations of system

(4.2), the closed autonomous system of differential equations:

$$\dot{y} = -\frac{\varepsilon\Delta}{(1-e^2)^{9/2}} \cdot \frac{(1-y^2)}{\omega_0} \cdot \left\{ \omega_0 \cdot y \cdot F_1(e) - \frac{2F_2(e)}{(1-e^2)^{3/2}} \right\},$$
$$\dot{\omega}_0 = -\frac{\varepsilon\Delta}{(1-e^2)^{9/2}} \cdot \left\{ \omega_0 \cdot (1+y^2) \cdot F_1(e) - \frac{2yF_2(e)}{(1-e^2)^{3/2}} \right\}, \quad (4.3)$$

where $\Delta = 9\chi\rho_1^2 A^{-1} D n^4$.

System (4.3) has an asymptotically stable stationary solution:

$$y = 1, \ \omega_0^* = \frac{F_2(e)}{F_1(e) \cdot (1-e^2)^{3/2}}. \quad (4.4)$$

In the stationary motion, the kinetic momentum vector **G** is orthogonal to the plane of the orbit, and the limiting value of the angular velocity of the proper rotation depends on the eccentricity of the elliptical orbit. Stationary solution (4.4) was previously obtained by Beletskii [2, 3], who modeled a planet by a rigid body and represented the tidal torque by a phenomenological formula.

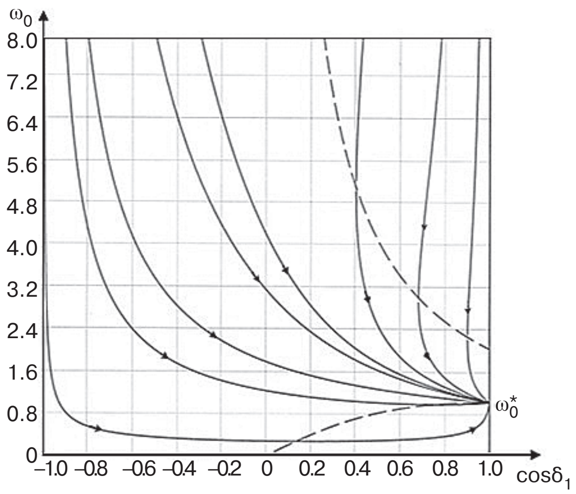


**Fig. 2.** Phase portrait of the evolutionary system of equations of motion

Figure 2 presents the phase portrait of system (4.3), which was constructed in the Octave environment at $e = 0.05$. The dashed lines are the loci of points at which the integral curves have horizontal and vertical tangent lines. All the integral curves collapse to the same point $(1, \omega_0^*)$. One can distinguish three types of motion: (1) monotonic decrease in the dimensionless angular velocity to a stationary value and monotonic decrease in the angle $\delta_1$ to zero (when the integral curves do not intersect the dashed lines); at points of the straight line $\cos\delta_1 = 0$, there is a transition from the reverse to the direct rotation; (2) monotonic decrease in the angle $\delta_1$ to zero and monotonic decrease in the dimensionless angular velocity to a certain minimum value with subsequent increase to a stationary value (when the integral curves intersect the lower dashed line); and (3) monotonic decrease in the dimensionless angular velocity to a stationary value and monotonic increase in the angle $\delta_1$ to a certain minimum value with subsequent decrease to zero (when the integral curves intersect the upper dashed line).

Previously [15], the methods of separation of motions and averaging were used to study the rotational motion of a satellite with flexible viscoelastic rods in an elliptical orbit.

## CONCLUSIONS

In this work, a study was made of the rotational motion of a planet modeled by a body comprising a rigid core and a viscoelastic shell attached rigidly to the core. A system of equations of motion was obtained within the linear theory of elasticity as a system of integro-partial differential equations in the form of Routh equations using the canonical Andoyer variables. The asymptotical method of separation of motions was used to derive a system of sixth-order ordinary differential equations describing the dynamics of the rotational motion of the planet. The averaging method was used to obtain an evolutionary system of equations of motion of the planet in the nonresonance case. It was shown that the motion of the planet tends to a stationary motion, in which the kinetic momentum vector **G** is orthogonal to the plane of the orbit, and its magnitude has a constant value depending on the eccentricity of the elliptical orbit. At zero eccentricity in the stationary motion, the angular velocity of the proper rotation of the planet coincides with the orbital angular velocity, and the axis of rotation of the planet is orthogonal to the plane of the orbit.

The results of this work can be used to investigate the tidal effects of the rotational motion of planets and their satellites.

**Authors' contribution.** All authors equally contributed to the research work.

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