Mathematical modeling

Математическое моделирование

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RESEARCH ARTICLE

Features of analytical modeling of nonlinear surface waves in gradient media

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Abstract

Objectives. An important role in modern physics, particularly in waveguide optics, is played by studies that involve the search for exact solutions to equations used in modeling to identify classes of exactly solvable models. This work set out to use analytical modeling methods to explore the properties of surface shear waves propagating without loss along the interface between a nonlinear and a graded-index nonmagnetic medium.

Methods. Methods of mathematical modeling, calculus, mathematical physics, differential equations, and the theory of special functions were used. Fundamental principles, methods, and physical models of nonlinear and waveguide optics were also applied.

Results. The properties of surface transverse waves propagating along the interface between a nonlinear and a graded-index medium are modeled. In order to model the nonlinearity of the medium to describe the nonlinear optical response of the medium to electric field perturbations, the linear dependence of permittivity on light intensity is chosen as a parameter. The graded-index medium is modeled using a spatial profile of permittivity as a function of distance from the interface for which an exact analytical solution to the stationary wave equation can be found. A mathematical formulation of the model is presented, consisting of a conjugation boundary value problem for a nonlinear equation with variable coefficients. Exact analytical solutions to this boundary value problem are found for the cases of focusing and defocusing nonlinearities to describe the spatial distributions of the electric field strength in the direction transverse to the interface. Analysis of the model revealed significant differences in the spatial distribution of the field intensity in surface waves propagating in the focusing and defocusing media. The effect of the values of model parameters used to characterize the optical properties of the contacting media on the spatial distribution of light intensity in surface waves was also studied in detail.

Conclusions. The obtained, which results supplement the existing theory of nonlinear and waveguide optics, can be applied in the design of new waveguide structures with user-defined properties. The obtained new solutions expand the class of exactly solvable models of planar waveguide structures with distributed inhomogeneous and nonlinear properties.

Keywords: mathematical modeling, mathematical model, boundary value problem, exact solution, surface wave, waveguide optics, nonlinear optics

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НАУЧНАЯ СТАТЬЯ

Особенности аналитического моделирования нелинейных поверхностных волн в градиентных средах

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Резюме

Цели. Важную роль в современной физике и волноводной оптике играют и исследования, связанные с нахождением точных решений используемых при моделировании уравнений, позволяющие выявить классы точно решаемых моделей. Цель работы – изучение свойств поверхностных поперечных волн, распространяющихся вдоль границы раздела нелинейной и градиентной немагнитных сред без потерь.

Методы. В работе использованы методы математического моделирования, методы анализа и математической физики, дифференциальных уравнений и теории специальных функций. Использовались базовые принципы, методы и физические модели нелинейной и волноводной оптики.

Результаты. Проведено моделирование свойств поверхностных поперечных волн, распространяющихся вдоль границы раздела нелинейной и градиентной сред. В качестве модели нелинейности среды, описывающей нелинейно-оптический отклик среды на возмущения электрического поля, выбрана линейная форма зависимости диэлектрической проницаемости от интенсивности света. В качестве модели градиентной среды выбрана форма пространственного профиля диэлектрической проницаемости, описывающая ее изменение в зависимости от расстояния до границы раздела, для которой можно найти точное аналитическое решение стационарного волнового уравнения. Приведена математическая формулировка модели, которая представляет собой краевую задачу сопряжения для нелинейного уравнения с переменными коэффициентами. Найдены точные аналитические решения данной краевой задачи для случаев фокусирующей и дефокусирующей нелинейностей, которые описывают пространственные распределения напряженности электрического поля в поперечном границе раздела сред направлении. Анализ модели позволил выявить существенные различия пространственного распределения интенсивности поля в поверхностных волнах, распространяющихся в фокусирующих и дефокусирующих средах. Проведен детальный анализ влияния значений параметров модели, характеризующих оптические свойства контактирующих сред, на пространственное распределение интенсивности света в поверхностных волнах.

Выводы. Полученные результаты дополняют существующую теорию нелинейной и волноводной оптики и могут найти применение при проектировании новых волноводных структур с определяемыми пользователями свойствами. Полученные новые решения расширяют класс точно решаемых моделей планарных волноводных структур с распределенными неоднородными и нелинейными свойствами.

Ключевые слова: математическое моделирование, математическая модель, краевая задача, точное решение, поверхностная волна, волноводная оптика, нелинейная оптика

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INTRODUCTION

Mathematical modeling occupies a central place in modern physics and waveguide optics. In particular [1], this approach can be used to identify, describe, and predict the behavior of a physical system under the influence of various parameters. Most mathematical models of physical processes are based on boundary value problems for partial differential equations, which under certain conditions can transform into ordinary differential equations. Modern computing systems can be used not only to find solutions to such problems using numerical methods but also to visualize them. However, an important role in applied mathematical research his also played by problems of finding exact solutions to formulated equations, which permit the identification of classes of exactly solvable models. The existence of an exact solution, as opposed to a numerical one, makes it possible not only to quickly and clearly analyze the effect of model parameters on the process under study, but also to obtain asymptotic expressions and other estimates in explicit analytical form [2].

The obtaining of exact solutions to boundary value problems arising in the modeling of surface wave propagation in planar waveguide structures is discussed in the work [3]. Although waveguide properties of interfaces between media with different optical characteristics have been examined in numerous works [4–6], many unexplored problems remain that involve the analytical description of surface wave properties within the framework of waveguide models that allow exact solutions. Such models are referred to as exactly solvable [7].

In nonlinear and waveguide optics, the fundamental model equation is the wave equation, which is derived from Maxwell's equations. For time-independent wave processes, the wave equation transforms into the Helmholtz equation [8]. The optical properties of the medium in which the wave process is being studied are determined by the refractive index or permittivity (for nonmagnetic media). Therefore, the choice of the form of dependence of these characteristics on the process function (e.g., light intensity) or spatial variables determines the possibility of an exact solution to the model equation. Such dependencies, which are themselves model equations, characterize the optical response of the medium to various perturbations.

The nonlinear response of a medium is described by models in which the permittivity depends on the light intensity, i.e., on the square of the amplitude of the electric (or magnetic) field strength [9]. Many of these models allow exact analytical solutions. The most common of these is the Kerr nonlinearity model, in which the permittivity depends linearly on the light intensity. In such a nonlinear model, depending on the geometry of the system, there are various classes of exact solutions for nonlinear equations, known as solitons [10–12], kinks, and cnoidal waves [13]. Furthermore [12], solitons can be not only optical, but also of a different physical nature, which emphasizes the wide application of similar mathematical models fora large class of phenomena that differ in physical nature.

The spatial inhomogeneity of the optical properties of a medium is characterized using models in which the permittivity depends on spatial coordinates. In waveguide optics, the media in which the refractive index smoothly depends on the spatial coordinate are referred to as graded-index media [14]. In this case, the equation contains variable (distributed) coefficients determined by the chosen inhomogeneity model. Exact analytical solutions are known for a wide range of models; these solutions are expressed through various special functions [15].

From a physical perspective, it is important to investigate the characteristics of wave propagation along interfaces between media having different optical properties. When modeling such processes, boundary value problems with conjugation conditions at the interface are formulated. A number of studies examined various surface waves propagating along interfaces between nonlinear and inhomogeneous media [16]. Exact solutions to model conjugation boundary value problems describing various types of surface waves were obtained [17, 18].

Recently [19], a study was made of surface waves along the surface of an optically inhomogeneous crystal with permittivity ε modeled by a special type of profile generalizing a hyperbolic profile ($\varepsilon \sim 1/x$) [20] and an inverse square profile ($\varepsilon \sim 1/x^2$) [21]. Consideration was made of the contact of an inhomogeneous medium having such a profile with a homogeneous linear medium in which the field intensity decreases exponentially with distance from the interface between the media. An exact analytical solution to the posed conjugation boundary value problem was found, the possibility of whose expression through the Whittaker function has been demonstrated.

The present work examines the contact of an inhomogeneous medium with an optical nonlinear medium having the same spatial profile. Kerr nonlinearity was chosen as the model of the nonlinear response. It is confirmed that the boundary value problem has an exact solution with this choice of inhomogeneity and nonlinearity models. The resulting solution is used to identify the characteristics of the formation of the spatial profile of the electric field in the transverse direction of the interface between the media at various optical parameters of the model.

The main result of this work is the derivation of an exact solution for this configuration of media. This solution describes a new type of wave with a novel shape of its profile in the direction transverse to the interface between the media. Furthermore, the sensitivity of the properties of this wave to changes in the optical parameters of the media is revealed.

1. MATHEMATICAL FORMULATION OF THE MODEL

Consider a plane interface between two lossless nonmagnetic media having different optical properties. Let the interface plane be located in the $x = 0$ coordinate plane (the yOz plane in space), and let the x axis be perpendicular to the interface and aligned with the direction of surface wave propagation. If the media are assumed to be uniform along their interface, then the electric field distribution in the transverse wave will also be uniform along it and characterized by spatial differences only in the transverse direction. When modeling surface waves in such systems, the problem is reduced to a one-dimensional one in the direction perpendicular to the interface; its solution describes the spatial distribution of the y component of the electric field strength vector in this direction.

As is known [8], the fundamental equation of the theory of waveguide optics for the transverse distribution $u(x)$ of the electric field in a surface wave can be written in the form

$$u''(x) + \{k_0^2 \varepsilon(x, I) - \beta^2\} u(x) = 0, \tag{1}$$

where $\varepsilon(x, I)$ is the permittivity; $I = u^2$ is the intensity of the electric field (light); $k_0 = 2\pi/\lambda_0$ is the longitudinal wave number; λ_0 is the wavelength of the sent radiation; and β is the propagation constant, which is related to the effective refractive index n by the expression $\beta = nk_0$.

Equation (1) is, in general, a nonlinear equation with variable coefficients.

The optical properties of media and their spatial distribution are completely described by the permittivity $\varepsilon(x, I)$. Therefore, the differences in models of media are described by this function. In the case of a flat interface between media, such a function is represented in step form:

$$\varepsilon(x, I) = \begin{cases} \varepsilon_G(x), & x > 0, \\ \varepsilon_N(I), & x < 0, \end{cases} \tag{2}$$

where it is considered that the model describes the contact of a nonlinear medium with a graded-index medium and assumed that the nonlinear medium occupies a half-space $x < 0$, while the graded-index medium fills the half-space $x > 0$.

In (2), the function $\varepsilon_N(I)$ describes the dependence of the permittivity on the light intensity in the nonlinear medium. Its specific form is determined by the model of the medium's nonlinearity, which characterizes the nonlinear optical response of the medium to electric field perturbations arising from the redistribution of internal charges with varying intensity of the laser radiation exciting the surface wave. Note that there is a wide variety of real materials exhibiting Kerr nonlinearity. For example, these include crystals like $AgGaSe_xS_{2(1+x)}$, which have high nonlinearity, including Kerr nonlinearity, and in which the refractive index changes proportionally to the square of the light field strength, as well as GaP, InAs, InP, and InSb semiconductors.

The function $\varepsilon_G(I)$ actually characterizes the spatial inhomogeneity of the refractive index profile in a graded-index medium. Its specific form is determined by the graded-index medium model, in which the change in refractive index with distance from the surface depends on the spatial distribution of charges induced, for example, by implanted ions. In this case, the refractive index profile can be similar to the spatial distribution of the implanted ion concentration. This means that, if it is technologically possible to create a specific ion concentration profile by implantation, a corresponding refractive index profile will be obtained to determine the function $\varepsilon_G(I)$.

In this paper, we consider the most common model of nonlinearity in the form of a linear dependence of permittivity on light intensity (i.e., a quadratic dependence on electric field). This model describes nonlinear media that exhibit the Kerr effect and are called Kerr media. In the case of Kerr nonlinearity, the function $\varepsilon_N(I)$ can be written as

$$\varepsilon_N(I) = \varepsilon_{0N}(I) + \alpha I, \quad (3)$$

where α is the Kerr nonlinearity coefficient, and $\varepsilon_{0N} = \varepsilon_N(I = 0)$ is the unperturbed value of the permittivity of the nonlinear medium.

In nonlinear optics, a positive value of the Kerr nonlinearity coefficient is characteristic of media in which a self-focusing effect of light beam propagation is exhibited (focusing media), while a negative value models media characterized by a defocusing effect (defocusing media) [9]. The choice of the nonlinearity model in form (3) is primarily due to the fact that Eq. (1) with coefficient (3) has exact analytical solutions, the specific form of which is determined by the sign of the Kerr nonlinearity coefficient and additional conditions (boundedness, periodicity, etc.).

To model the spatial inhomogeneity of the refractive index in the graded-index medium in this work, we use a function of the form

$$\varepsilon_G(x) = e_0 + \frac{e_1}{x+h} + \frac{e_2}{(x+h)^2}, \quad (4)$$

where e_0, e_1, e_2, and h are the parameters of the spatial profile of the permittivity.

The choice of the graded-index medium model in form (4) is primarily due to the fact that Eq. (1) with coefficient (4) has exact analytical solutions [19], whose specific form is determined by the profile parameters and expressed through special functions of mathematical physics. In addition, the shape of the permittivity profile of form (4) is a generalization of the monotonically decreasing profiles of the refractive index that were previously used in modeling surface waves propagating along the contact of graded-index media with a Kerr nonlinear medium. In particular, at $e_0 = e_2 = 0$, expression (4) takes the form of the profile $\varepsilon_G(x) = e_1/(x + h)$ [20], and at $e_0 = e_1 = 0$, Eq. (4) becomes the profile $\varepsilon_G(x) = e_2/(x + h)^2$ [21].

A refractive index profile that sharply monotonically (but not exponentially) decreases within a narrow region was mentioned in the development of special photonic crystal heterostructures [22, 23]. A profile of form (4) is a more general version of the hyperbolic profile mentioned in those works. Selecting the parameters of profile (4) allows for a more accurate approximation of experimentally obtained profiles of a similar class. In this regard, interest arises in theoretically studying the possibility of constructing a crystalline system as a composite of the above types of optical materials, which could lead to the discovery of new properties.

Since the coefficient ε in Eq. (1) is generally discontinuous (with a jump), its solution can be represented as

$$u(x) = \begin{cases} u_G(x), & x > 0 \\ u_N(I), & x < 0, \end{cases} \quad (5)$$

where the sought functions $u_G(x)$ and $u_N(x)$ describe the spatial transverse distributions of the electric field strength in the graded-index and nonlinear media, respectively; and are defined on the corresponding half-axes.

As a result, instead of Eq. (1), taking into account (2)–(5), two equations can be written on the half-axes:

$$u''_G(x) + \left(e_0 + \frac{e_1}{x+h} + \frac{e_2}{(x+h)^2} - n^2\right) \times \\ \times k_0^2 u_G(x) = 0, \; x > 0, \quad (6)$$

$$u''_N(x) + \{\varepsilon_{0N} + \alpha |u_N|^2 - n^2\} k_0^2 u_N(x) = 0, \; x < 0. \quad (7)$$

The requirement for continuity of the components of the electromagnetic field when moving from one of the media to the other leads to the need to use boundary conditions of conjugation at the interface between the media at $x = 0$:

$$u_G(+0) = u_G(-0), \quad (8)$$

$$u'_G(+0) = u'_N(-0). \quad (9)$$

From here on, we mean one-sided limits $\lim_{x \to \pm 0} f(x) = f(\pm 0)$.

Since the field must tend to zero at infinity, the following conditions at infinity naturally arise:

$$\lim_{x \to \pm\infty} u_{G,N}(x) = 0. \quad (10)$$

Thus, the mathematical formulation of the model is a boundary value problem of finding continuous and bounded solutions of Eqs. (6) and (7) on the corresponding half-axes that are related by conjugation conditions (8) and (9) and satisfy the conditions at infinity (10).

2. ANALYTICAL RESULTS

Equation (6) is an equation with variable coefficients, and its solution bounded on the positive half-axis can be written in various ways: through a hypergeometric function, a confluent Heun function, or also a Whittaker function. In our opinion, the preferred form of representing the solution is through the Whittaker function.

A solution, bounded on the positive half-axis, to the Whittaker differential equation [24]

$$y'' + \left(\frac{\mu}{z} - \frac{1}{4} + \frac{1/4 - \nu^2}{z^2}\right) y = 0$$

is the Whittaker function $W_{\mu,\nu}(z)$. By changing the variables, Eq. (6) is reduced to the Whittaker equation, and then its solution bounded on the positive half-axis can be written as

$$u_G(x) = u_0 \frac{W_{\mu,\nu}\left(p(x+h)\right)}{W_{\mu,\nu}(ph)}, \tag{11}$$

where u_0 is the amplitude of the electric field strength at the interface between the media, and the subscripts and parameters of the Whittaker function are determined by the coefficients of Eq. (6) as

$$\mu = \frac{e_1 k_0}{2\sqrt{n^2 - e_0}}, \tag{12}$$

$$\nu = \sqrt{1 - 4k_0^2 e_2}\ /\ 2, \tag{13}$$

$$p = 2k_0\sqrt{n^2 - e_0}. \tag{14}$$

For such a solution to exist, the conditions $n^2 > e_0$ and $k_0^2 > 1/4e_2$ must be met. Due to the choice of the Whittaker function, solution (11) satisfies the condition at infinity (10), which for it has the form $u_G(x) \to 0$ as $x \to +\infty$.

Equation (7) is a nonlinear differential equation. It has several types of solutions, depending on the sign of the nonlinearity coefficient α. Therefore, further analysis of the model is carried out separately for the cases $\alpha > 0$ and $\alpha < 0$.

(1) The case of focusing nonlinearity ($\alpha > 0$).

At $\alpha > 0$, a solution, bounded on the negative half-axis, to Eq. (7) satisfies the condition at infinity (10), which for it has the form $u_N(x) \to 0$ as $x \to -\infty$, and is expressed through the hyperbolic cosine:

$$u_N(x) = \sqrt{\frac{2}{\alpha}} \cdot \frac{q}{k_0 \cosh\left(q(x - x_N)\right)}. \tag{15}$$

Here,

$$q^2 = k^2(n^2 - \varepsilon_{0N}), \tag{16}$$

and the quantity x_N is the position of the maximum (if any) of the intensity of the surface wave in the nonlinear medium, which is determined from the boundary conditions.

To determine the field amplitude at the interface and x_N, solutions (11) and (15) should be substituted into boundary conditions (8) and (9). Such transformations give

$$u_0 = \sqrt{\frac{2}{\alpha}} \cdot \frac{q}{k_0 \cosh(q x_N)}, \tag{17}$$

$$x_N = \frac{1}{q}\operatorname{artanh}\left(\frac{q_G}{q}\right), \tag{18}$$

where

$$q_G = p\frac{W'_{\mu,\nu}(ph)}{W_{\mu,\nu}(ph)}. \tag{19}$$

Taking into account expression (18), from relation (17), we can obtain the field intensity at the interface between the media in the form

$$I_0 = \left|u_0\right|^2 = \frac{2}{\alpha}(n^2 - \varepsilon_{0N} - q_G^2 / k_0^2). \tag{20}$$

Thus, an everywhere continuous, smooth, bounded, vanishing at infinity solution to the posed boundary value problem in the case of positive nonlinearity can be written in the form

$$u(x) = \sqrt{\frac{2}{\alpha}} \cdot \begin{cases} \dfrac{W_{\mu,\nu}\left(p(x+h)\right)}{W_{\mu,\nu}(ph)}, & x > 0, \\ \dfrac{q}{k_0 \cosh\left(q(x - x_N)\right)}, & x < 0. \end{cases} \tag{21}$$

Expression (21) describes a nonlinear surface wave propagating along the interface between the nonlinear focusing and the graded-index media using the selected models of nonlinearity and spatial distribution of the refractive index.

(2) The case of defocusing nonlinearity ($\alpha < 0$).

At $\alpha < 0$, a solution, bounded on the negative half-axis, to equation (7) satisfies the condition at infinity (10), which for it has the form $u_N(x) \to 0$ as $x \to -\infty$, and is expressed through the hyperbolic sine:

$$u_N(x) = -\sqrt{\frac{2}{|\alpha|}} \cdot \frac{q}{k_0 \sinh\left(q(x - x_N)\right)}. \tag{22}$$

To determine the field amplitude at the interface and x_N, solutions (11) and (22) should be substituted into boundary conditions (8) and (9). Such transformations give

$$u_0 = \sqrt{\frac{2}{\alpha}} \cdot \frac{q}{k_0 \sinh(q x_N)}, \tag{23}$$

$$x_N = \frac{1}{q}\operatorname{arcoth}\left(\frac{q_G}{q}\right). \tag{24}$$

Taking into account (24), from (23), we can obtain the field intensity at the interface between the media in the form

$$I_0 = \frac{2}{|\alpha|}(q_G^2 / k_0^2 + \varepsilon_{0N} - n^2). \tag{25}$$

Thus, an everywhere continuous, smooth, bounded, vanishing at infinity solution of the posed boundary value problem in the case of negative nonlinearity can be written in the form

$$u(x)=\sqrt{\frac{2}{|\alpha|}}\cdot\begin{cases}\dfrac{W_{\mu,\nu}\left(p(x+h)\right)}{W_{\mu,\nu}(ph)}, & x>0,\\[2ex] \dfrac{q}{k_0\sinh\left(q(x-x_N)\right)}, & x<0.\end{cases} \quad (26)$$

Expression (26) describes a nonlinear surface wave propagating along the interface between the nonlinear defocusing and graded-index media using the selected models of nonlinearity and spatial distribution of the refractive index.

3. RESULTS OF MODELING THE PROPERTIES OF SURFACE WAVES

First of all, we note the differences in the spatial distribution of the field intensity *I* in the surface waves propagating in the focusing and defocusing media in contact with the graded-index medium in the considered heterogeneity model.

Figure 1 shows the characteristic transverse intensity profiles constructed using solutions (21) and (26) at the same values of the optical parameters. Note that we chose $|\alpha| = 1$, specifically, $\alpha = 1$ for the case of the focusing medium (line *1*) and $\alpha = -1$ for the case of the defocusing medium (line *2*). All other parameter values are the same, and they are chosen so that their values fall within the ranges of existence of both surface waves.

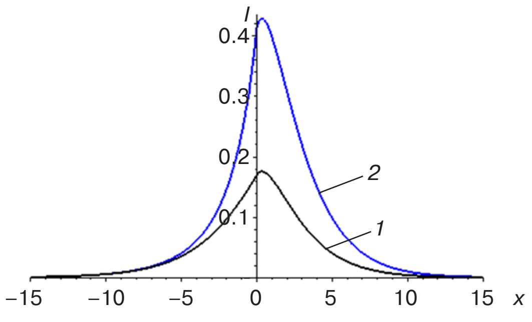

Fig. 1. Spatial distributions of the field intensity in the surface waves determined by expressions (21) (line *1*) and (26) (line *2*) at the following values of the system parameters (in conventional dimensionless units): $k_0 = 0.5$, $h = 0.5$, $e_0 = -0.1$, $e_1 = 0.8$, $e_2 = 0.2$, $\varepsilon_{0N} = 0.05$, $n = 0.44$, and α = (*1*) 1 and (*2*) −1

The surface waves are characterized by a single, clearly defined maximum in the spatial distribution of the field intensity, which decreases monotonically with distance from the maximum in both directions from the interface. In the selected range of parameter values, the maximum of the intensity for both types of surface waves is located in the graded-index medium. As will be shown below, in the case of defocusing nonlinearity, the maximum of the intensity is always located in the graded-index medium, while in the case of focusing nonlinearity, it can also be located in the nonlinear medium (not only in the graded-index medium), but at other parameter values. In other words, in the defocusing nonlinear medium, the field always decays strictly monotonically at any parameter values. In the focusing medium, the decay can be nonmonotonic, with an intensity spike near the interface, in which case the decay in the graded-index medium becomes monotonic.

The intensity peak height in the surface wave in the defocusing medium (Fig. 1, line *2*) significantly exceeds the intensity peak height in the surface wave in the focusing medium (Fig. 1, line *1*). For example, at the parameter values selected in Fig. 1, this excess is more than twofold. This results in a higher light intensity in the surface wave in the defocusing medium in comparison with the intensity in the surface wave in the focusing medium at the same distance from the interface.

Below, we will present the results of modeling the properties of the surface waves due to changes in the optical characteristics of the media. For the cases of focusing and defocusing media, this analysis is performed separately.

Note that varying the parameters of the models of nonlinearity (3) and graded-index media (4) means that different media correspond to different parameter values. Varying the effective refractive index *n* (which is equivalent to varying the propagation constant β) means changing the angle of incidence of the beam exciting the surface wave in a given medium. Varying the longitudinal wavenumber k_0 means changing the wavelength of the radiation exciting the surface wave (for example, the wavelength of a laser).

(1) The case of focusing nonlinearity ($\alpha > 0$).

Modeling using analytical solution (21) revealed that, depending on the effective refractive index, the surface wave propagating along the interface with the focusing nonlinear medium can be characterized by an intensity maximum located in both the nonlinear and the graded-index media. Specifically, at relatively low effective refractive index values, the intensity maximum is located in the graded-index medium. As the effective refractive index increases, the maximum shifts closer to the interface and then transitions into the nonlinear medium (Fig. 2), while its height also increases. The field penetration depth into the nonlinear medium increases; however, in the graded-index medium, it initially increases and then begins to decrease with increasing effective refractive index.

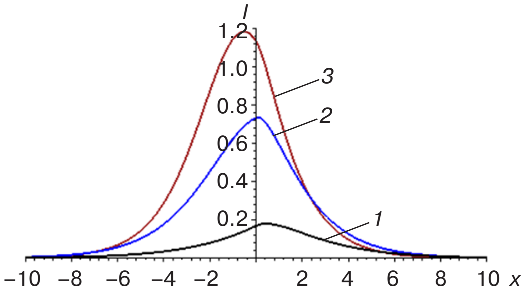

Fig. 2. Spatial distributions of the field intensity in the surface waves in the focusing medium (expression (21)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = 1$, $k_0 = 0.5$, $h = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.1$, $\varepsilon_{0N} = 0.05$; and n = (*1*) 0.5, (*2*) 0.65, and (*3*) 0.8

It can be seen that there are two distinct characteristic field intensity distributions in the surface wave propagating along the interface with the focusing nonlinear medium at two different values of the effective refractive index. One type corresponds to the maximum intensity in the graded-index medium (Fig. 2, line *1*), while the other corresponds to the maximum intensity in the nonlinear medium (Fig. 2, line *3*).

Next, we will analyze the change in the spatial distribution of the surface wave intensity at these two characteristic values of the effective refractive index when varying the parameters of the models of the media.

Figure 3 illustrates the effect of the parameter h of graded-index profile (4) on the spatial distribution of the surface wave intensity. With increasing h, the position of the intensity maximum shifts from the graded-index medium to the nonlinear medium. However, the shape of the field profiles depends on the effective refractive index. In particular, at small values of n (Fig. 3a), the field intensity is lower than that at its large values (Fig. 3b).

Figure 4 presents the effect of the parameter e_2 of graded-index profile (4) on the spatial distribution of the surface wave intensity. At small values of the effective refractive index (Fig. 4a), with increasing e_2, the position of the intensity maximum located in the graded-index medium remains unchanged, and its height decreases. At large values of the effective refractive index (Fig. 4b), with increasing e_2, the position of the intensity maximum located in the nonlinear medium shifts to the graded-index medium, and its height also decreases.

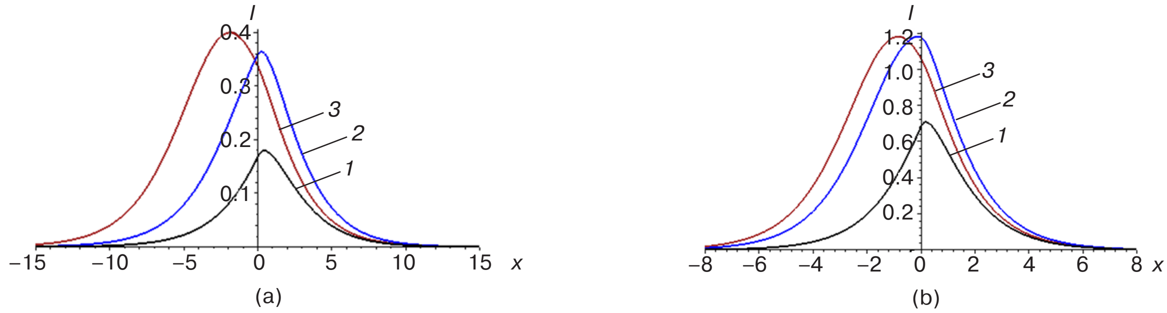

Fig. 3. Spatial distributions of the field intensity in the surface waves in the focusing medium (expression (21)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = 1$, $k_0 = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $\varepsilon_{0N} = 0.05$, (a) $n = 0.5$ and h = (*1*) 0.5, (*2*) 0.7, and (*3*) 1.5; and (b) $n = 0.8$ and h = (*1*) 0.2, (*2*) 0.4, and (*3*) 0.6

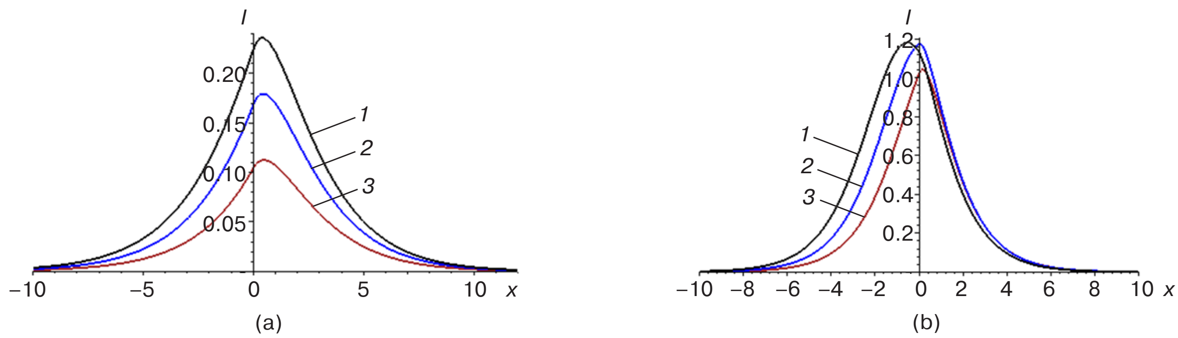

Fig. 4. Spatial distributions of the field intensity in the surface waves in the focusing medium (expression (21)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = 1$, $k_0 = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, (a) $n = 0.5$ and e_2 = (*1*) 0.15, (*2*) 0.2, and (*3*) 0.25; and (b) $n = 0.8$ and e_2 = (*1*) 0.2, (*2*) 0.5, and (*3*) 0.8

Figure 5 shows the effect of the parameter e_1 of graded-index profile (4) on the spatial distribution of the surface wave intensity. At small values of the effective refractive index (Fig. 5a), with increasing e_1, the position of the intensity maximum located in the graded-index medium shifts into the depth of the graded-index medium, and its height decreases. At large values of the effective refractive index (Fig. 5b), with increasing e_1, the position of the intensity maximum located in the nonlinear medium shifts into the graded-index medium, and its height also decreases.

Figure 6 displays the effect of the parameter e_0 of graded-index profile (4) on the spatial distribution of the surface wave intensity. At small values of the effective refractive index (Fig. 6a), with decreasing e_0, the position of the intensity maximum located in the graded-index medium shifts into the depth of the graded-index medium, and its height increases. At large values of the effective refractive index (Fig. 6b), with decreasing e_0, the position of the intensity maximum located in the nonlinear medium shifts into its depth, moving away from the interface between the media, and its height does not change.

Figure 7 demonstrates the effect of the parameter ε_{0N} of nonlinear model (3) on the spatial distribution of the surface wave intensity. At small values of the effective refractive index (Fig. 7a), with increasing ε_{0N}, the position of the intensity maximum located in the graded-index medium does not change, and its height decreases. At large values of the effective refractive index (Fig. 7b), a similar effect is observed when the position of the intensity maximum is located in the nonlinear medium, but the decrease in the height of the maximum is less significant than in the case of small n in the same range of ε_{0N}.

Figure 8 presents the effect of the nonlinearity coefficient α of model (3) on the spatial distribution of the surface wave intensity. At small (Fig. 8a) and large (Fig. 8b) values of the effective refractive index, with increasing α, the position of the intensity maximum does not change, and its height decreases. However, at large values of n, the penetration depth of the field into the nonlinear medium decreases more significantly than at small n.

Figure 9 demonstrates the effect of the optical parameters of the model on the intensity I_0 of the surface wave at the interface between the media (expression (20)). It should be noted that the dependencies of the intensity of the surface wave at the interface on the optical parameters

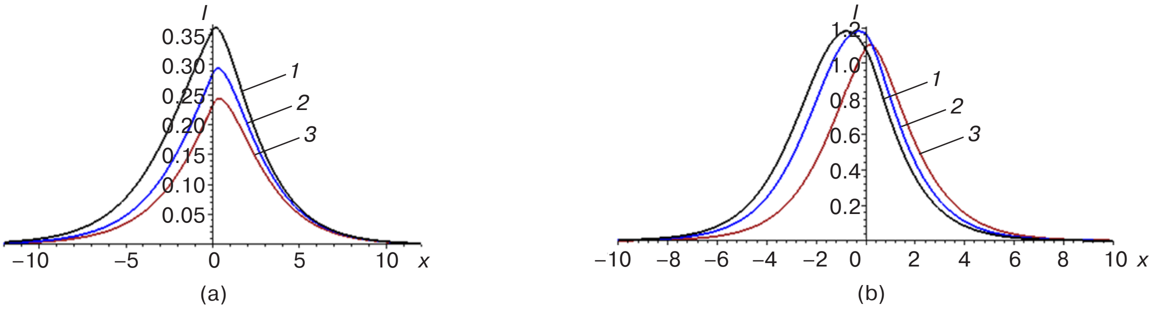

Fig. 5. Spatial distributions of the field intensity in the surface waves in the focusing medium (expression (21)) at the following values of the system parameters (in conventional dimensionless units):
$\alpha = 1$, $k_0 = 0.5$, $e_0 = -0.1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$,
(a) $n = 0.5$ and $e_1 =$ (*1*) 0.8, (*2*) 0.9, and (*3*) 0.95;
and (b) $n = 0.8$ and $e_1 =$ (*1*) 0.9, (*2*) 1.1, and (*3*) 1.5

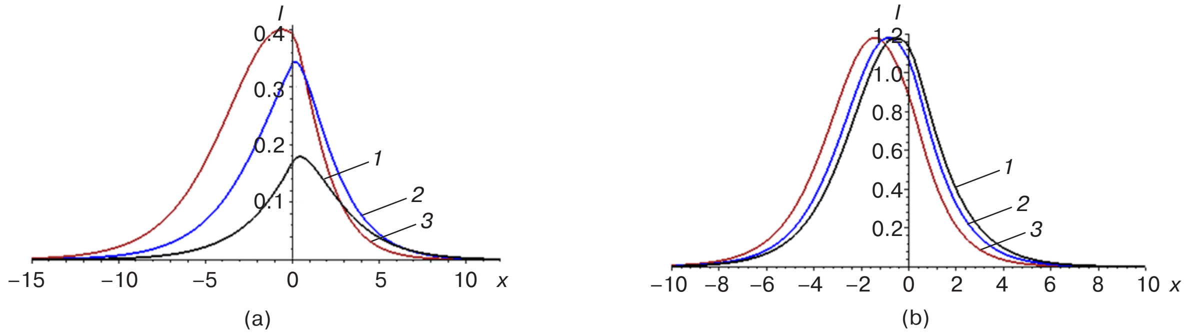

Fig. 6. Spatial distributions of the field intensity in the surface waves in the focusing medium (expression (21)) at the following values of the system parameters (in conventional dimensionless units):
$\alpha = 1$, $k_0 = 0.5$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$,
$e_0 =$ (*1*) −0.1, (*2*) −0.2, and (*3*) −0.4;
and $n =$ (a) 0.5 and (b) 0.8

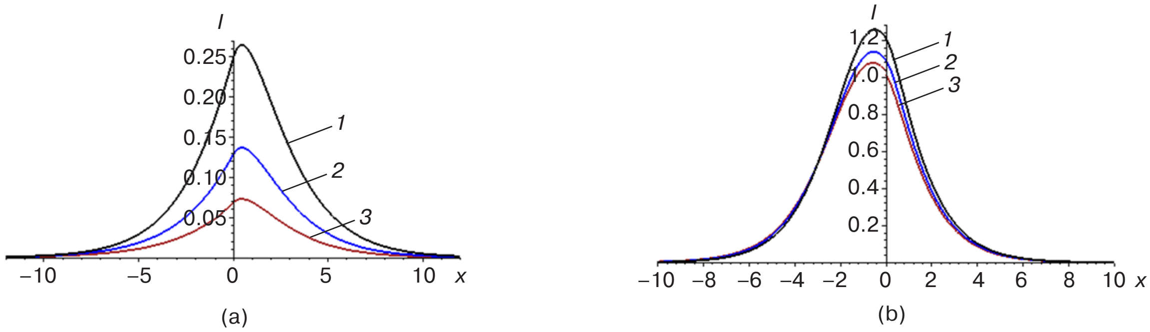

Fig. 7. Spatial distributions of the field intensity in the surface waves in the focusing medium (expression (21)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = 1$, $k_0 = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, ε_{0N} = (*1*) 0.01, (*2*) 0.07, and (*3*) 0.1; and n = (a) 0.5 and (b) 0.8

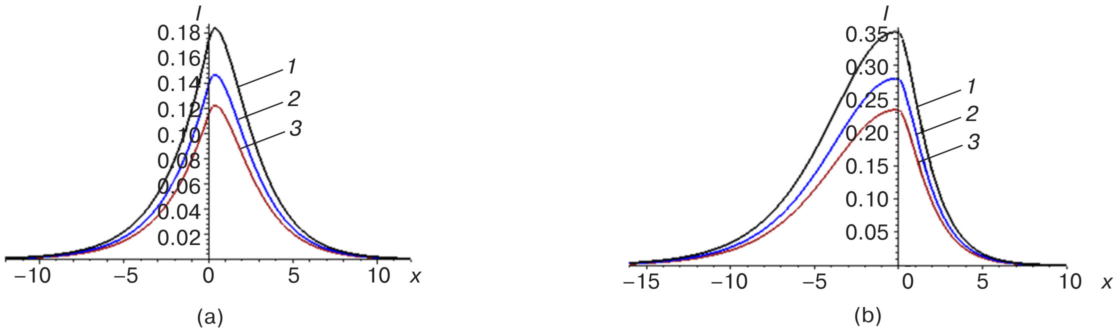

Fig. 8. Spatial distributions of the field intensity in the surface waves in the focusing medium (expression (21)) at the following values of the system parameters (in conventional dimensionless units): $k_0 = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, α = (*1*) 0.8, (*2*) 1, and (*3*) 1.2; and n = (a) 0.5 and (b) 0.8

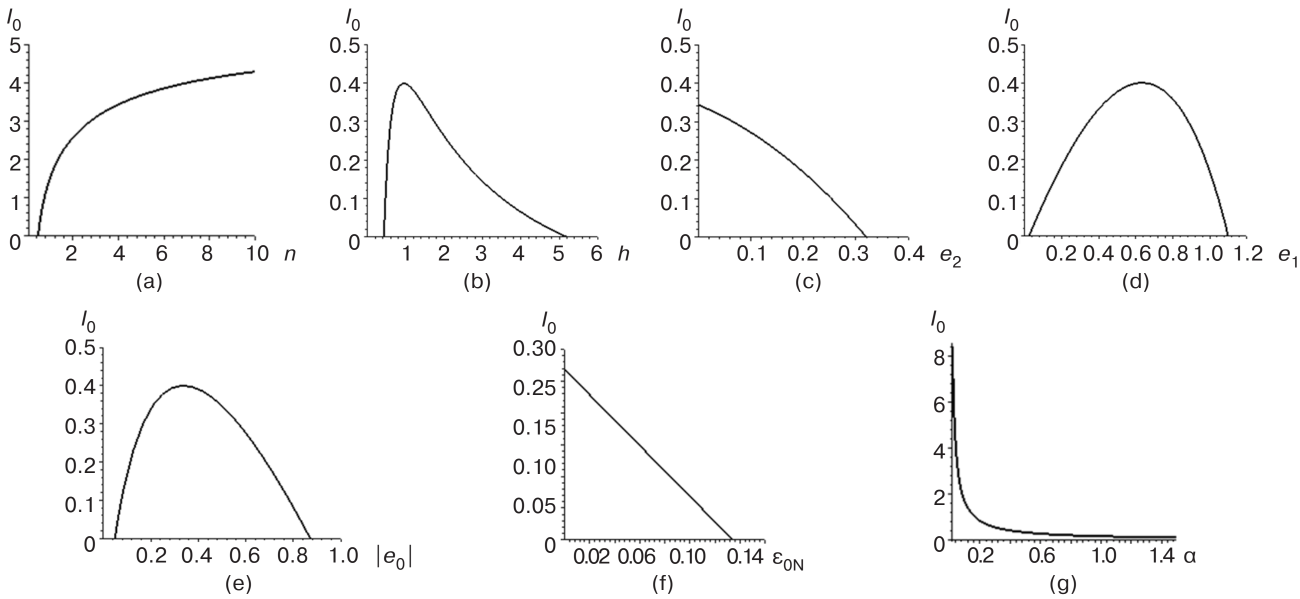

Fig. 9. Field intensity at the interface between the media (expression (20)) at the following values of the system parameters (in conventional dimensionless units): $k_0 = 0.5$;
(a) dependence on n at $h = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $\varepsilon_{0N} = 0.05$, and $\alpha = 1$;
(b) dependence on h at $n = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $\varepsilon_{0N} = 0.05$, and $\alpha = 1$;
(c) dependence on e_2 at $n = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, and $\alpha = 1$;
(d) dependence on e_1 at $n = 0.5$, $e_0 = -0.1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, and $\alpha = 1$;
(e) dependence on $|e_0|$ at $n = 0.5$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, and $\alpha = 1$;
(f) dependence on ε_{0N} at $n = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, and $\alpha = 1$; and
(g) dependence on α at $n = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, and $\varepsilon_{0N} = 0.05$

of the model are not all monotonic; i.e., at certain values of a number of optical parameters, I_0 has a maximum. Analysis of the modeling results showed that, with an increase in the effective refractive index, the intensity of the surface wave at the interface between the media increases monotonically (Fig. 9a). The intensity I_0 monotonically decreases with an increase in such optical parameters as e_2 (Fig. 9c), ε_{0N} (Fig. 9f), and α (Fig. 9g). Intensity maxima were found in the dependencies on such optical parameters as h (Fig. 9b), e_1 (Fig. 9d), and e_0 (Fig. 9e).

(2) The case of defocusing nonlinearity ($\alpha < 0$).

Modeling using analytical solution (26) revealed that the surface wave propagating along the interface with the defocusing nonlinear medium is always characterized by a single intensity maximum located in the graded-index medium. As the effective refractive index increases, its height decreases (Fig. 10), the field localization width decreases, and the position of the maximum shifts slightly toward the interface. This effect of the effective refractive index at the interface with the defocusing medium is opposite to the effect observed at the interface with the focusing medium (compare Figs. 3 and 10). Consequently, changing the angle of incidence of the beam exciting the surface wave has different (or, more accurately, opposite) effects on the field distribution in the surface waves in the focusing and defocusing media.

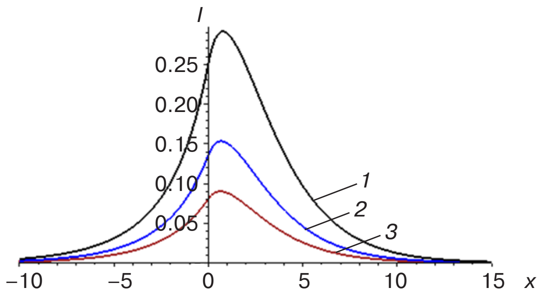

Fig. 10. Spatial distributions of the field intensity in the surface waves in the defocusing medium (expression (26)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = -1$, $k_0 = 0.5$, $h = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $\varepsilon_{0N} = 0.05$, and $n =$ (*1*) 0.42, (*2*) 0.44, and (*3*) 0.45

Figure 11 displays the effect of the parameter h of graded-index profile (4) on the spatial distribution of the intensity of the surface wave determined by solution (26). With increasing h, an effect similar to an increase in the effective refractive index is observed.

Figure 12 illustrates the effect of the parameter e_2 of graded-index profile (4) on the spatial distribution of the intensity of the surface wave determined by solution (26). With increasing e_2, the position of the intensity maximum located in the graded-index medium remains unchanged, its height increases, and the width of the field localization also increases.

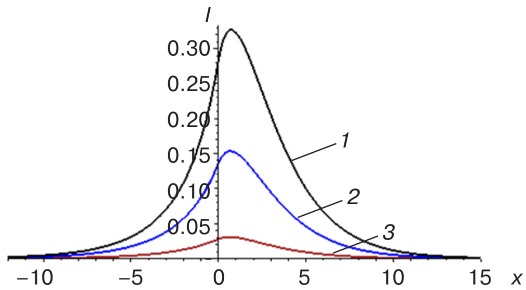

Fig. 11. Spatial distributions of the field intensity in the surface waves in the defocusing medium (expression (26)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = -1$, $k_0 = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $\varepsilon_{0N} = 0.05$, $n = 0.44$, and $h =$ (*1*) 0.45, (*2*) 0.50, and (*3*) 0.55

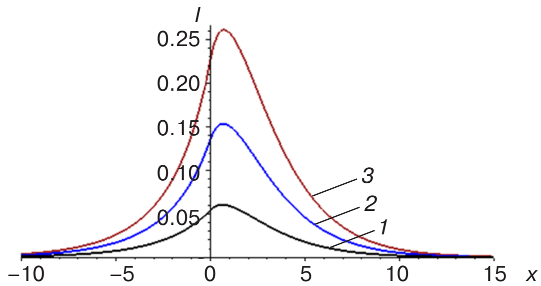

Fig. 12. Spatial distributions of the field intensity in the surface waves in the defocusing medium (expression (26)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = -1$, $k_0 = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, $n = 0.44$, and $e_2 =$ (*1*) 0.15, (*2*) 0.2, and (*3*) 0.25

Figure 13 presents the effect of the parameter e_1 of graded-index profile (4) on the spatial distribution of the intensity of the surface wave determined by solution (26). With increasing e_1, an effect similar to an increase in the parameter e_2 of graded-index profile (4) is observed.

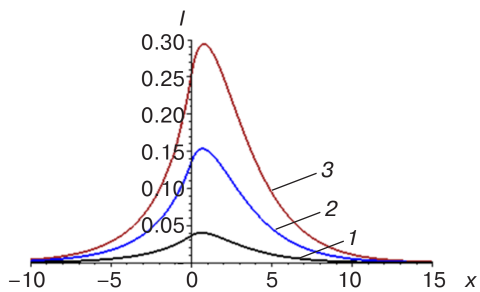

Fig. 13. Spatial distributions of the field intensity in the surface waves in the defocusing medium (expression (26)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = -1$, $k_0 = 0.5$, $e_0 = -0.1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, $n = 0.5$, and $e_1 =$ (*1*) 0.95, (*2*) 1.0, and (*3*) 1.05

Figure 14 demonstrates the effect of the parameter e_0 of graded-index profile (4) on the spatial distribution of the intensity of the surface wave determined by solution (26). With decreasing e_0, an effect similar to an increase in the effective refractive index and the parameter h is observed.

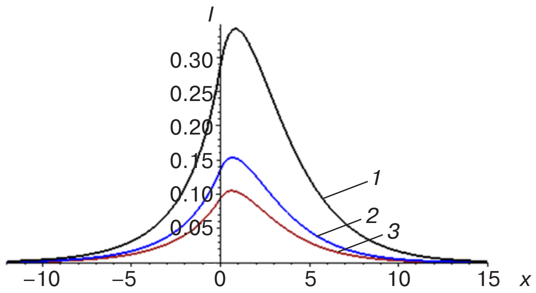

Fig. 14. Spatial distributions of the field intensity in the surface waves in the defocusing medium (expression (26)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = -1$, $k_0 = 0.5$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, $n = 0.44$, and e_0 = (*1*) −0.07, (*2*) −0.10, and (*3*) −0.11

Figure 15 shows the effect of the parameter ε_{0N} of nonlinear model (3) on the spatial distribution of the surface wave intensity determined by solution (26). With increasing ε_{0N}, an effect similar to an increase in the parameters e_1 and e_2 of graded-index profile (4) is observed.

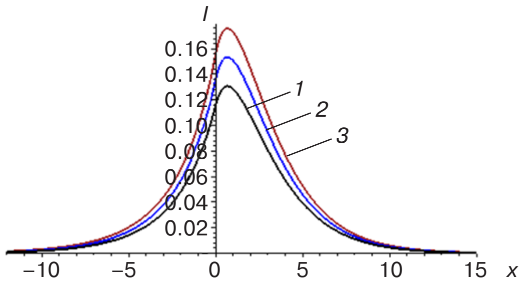

Fig. 15. Spatial distributions of the field intensity in the surface waves in the defocusing medium (expression (26)) at the following values of the system parameters (in conventional dimensionless units): $\alpha = -1$, $k_0 = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, $n = 0.44$, and ε_{0N} = (*1*) 0.04, (*2*) 0.05, and (*3*) 0.06

Figure 16 illustrates the effect of the nonlinearity coefficient α of model (3) on the spatial distribution of the intensity of the surface wave determined by solution (26). With a decrease in the Kerr nonlinearity coefficient (i.e., with an increase in the absolute value of α), an effect similar to an increase in the effective refractive index and the parameter h is observed.

Thus, it is clear that the effect of the optical parameters of the model on the profiles of the spatial intensity distribution in the surface waves propagating along the interfaces depends significantly on the sign of the nonlinearity coefficient, and the observed effects are often opposite in the focusing and defocusing media.

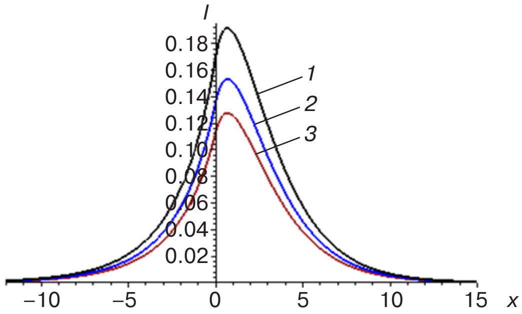

Fig. 16. Spatial distributions of the field intensity in the surface waves in the defocusing medium (expression (26)) at the following values of the system parameters (in conventional dimensionless units): $k_0 = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, $n = 0.44$, and α = (*1*) −0.8, (*2*) −1.0, and (*3*) −1.2

Figure 17 displays the effect of the optical parameters of the model on the intensity I_0 of the surface wave at the interface between the media (expression (25)). It should be noted that the dependencies of the intensity of the surface wave at the interface with the defocusing medium on the optical parameters of the model are all monotonic, unlike the case of contact with the focusing medium. Analysis of the modeling results showed that the surface wave intensity at the interface between the media decreases monotonically with increasing effective refractive index (Fig. 17a), parameter h (Fig. 17b), and absolute value of e_0 (Fig. 17e). The intensity I_0 increases monotonically with increasing optical parameters such as e_2 (Fig. 17c), e_1 (Fig. 17d), and ε_{0N} (Fig. 17f). In these cases, the effects observed in the defocusing medium are opposite to those observed in the focusing medium. The increase in the absolute value of the Kerr nonlinearity coefficient (Fig. 17g) is completely analogous to the increase in α in the focusing medium (Fig. 9g).

CONCLUSIONS

This paper presents the results of analytical modeling of surface shear waves propagating without loss along the interface between a nonlinear and a graded-index nonmagnetic medium. A linear dependence of permittivity on light intensity is chosen as a model of the nonlinearity of the medium to describe the nonlinear optical response of the medium to electric field perturbations. The graded-index medium is modeled using a special form of the spatial profile of permittivity as a function of distance to the interface for which an

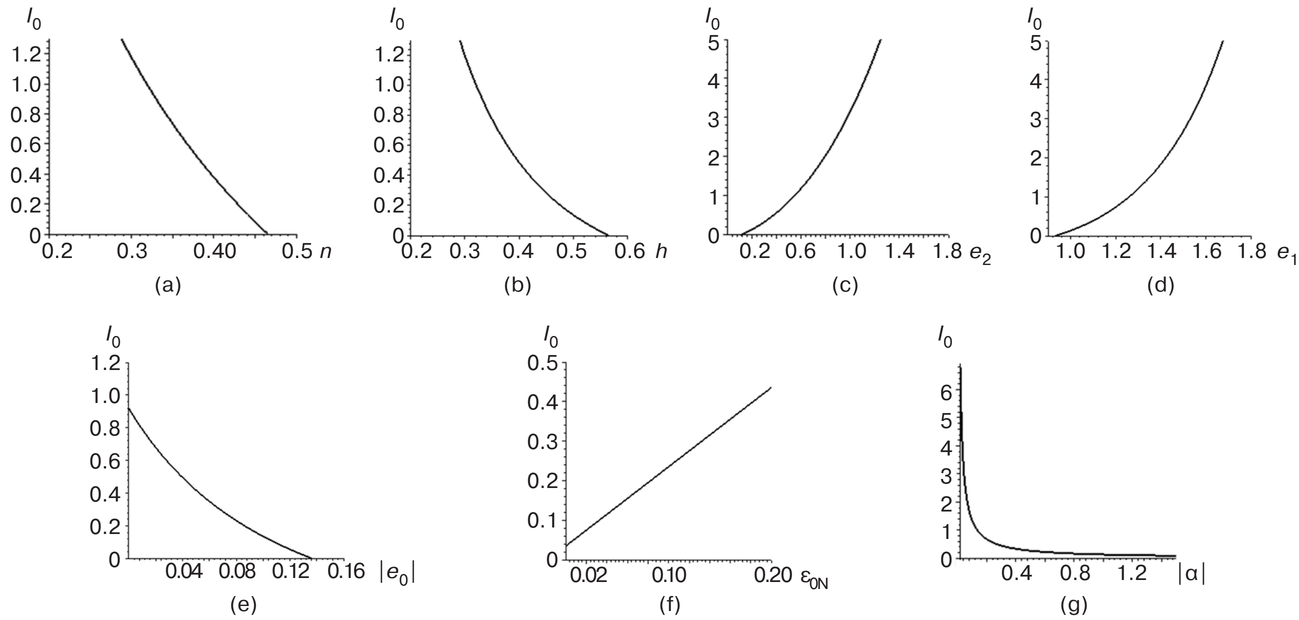

Fig. 17. Field intensity at the interface between the media (expression (25)) at the following values of the system parameters (in conventional dimensionless units): $k_0 = 0.5$;
(a) dependence on n at $h = 0.5$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $\varepsilon_{0N} = 0.05$, and $\alpha = -1$;
(b) dependence on h at $n = 0.44$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $\varepsilon_{0N} = 0.05$, and $\alpha = -1$;
(c) dependence on e_2 at $n = 0.44$, $e_0 = -0.1$, $e_1 = 1$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, and $\alpha = -1$;
(d) dependence on e_1 at $n = 0.44$, $e_0 = -0.1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, and $\alpha = -1$;
(e) dependence on $|e_0|$ at $n = 0.44$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, $\varepsilon_{0N} = 0.05$, and $\alpha = -1$;
(f) dependence on ε_{0N} at $n = 0.44$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, and $\alpha = -1$; and
(g) dependence on $|\alpha|$ at $n = 0.44$, $e_0 = -0.1$, $e_1 = 1$, $e_2 = 0.2$, $h = 0.5$, and $\varepsilon_{0N} = 0.05$

exact analytical solution of the stationary wave equation can be found.

The presented mathematical formulation of the model consists in a conjugation boundary value problem for a nonlinear equation with variable coefficients. Exact analytical solutions to this boundary value problem are found for the cases of focusing and defocusing nonlinearities to describe the spatial distributions of the electric field strength in the direction transverse to the interface.

Analysis of the model revealed significant differences in the spatial distribution of the field intensity in surface waves propagating in the focusing and defocusing media. In the case of defocusing nonlinearity, the intensity maximum is always located in the graded-index medium, while in the case of focusing nonlinearity, it can be located in either the graded-index or the nonlinear medium, but at different parameter values. Furthermore, the light intensity in the surface wave in the defocusing medium is higher than the intensity in the surface wave in the focusing medium at the same distance from the interface and the same model parameter values.

The effect of the values of model parameters characterizing the optical properties of the contacting media on the spatial distribution of the light intensity in the surface waves was analyzed in detail. The modeling results showed that changing the same parameters has different (or, more accurately, opposite) effects on the field distribution in the surface waves in the focusing and defocusing media. Specifically, as the effective refractive index increases, the height of the intensity distribution maximum increases in the focusing medium, while it decreases in the defocusing medium.

In this paper, we chose a different model of contacting optical media than the one considered in our previous studies. The resulting new analytical solutions differ from those obtained previously, leading to differences in the properties of the surface waves they describe, particularly with regard to the sensitivity of their profile shapes to changes in the optical parameters of the media.

The obtained results, which supplement the existing theory of nonlinear and waveguide optics, can be applied in the design of new waveguide structures whose required dispersion properties are determined by the intensity of surface waves and by the optical characteristics of nonlinear and graded-index media. The new solutions obtained expand the class of exactly solvable models of planar waveguide structures with distributed inhomogeneous and nonlinear properties.

Authors' contributions

S.E. Savotchenko—conceptualization, methodology, analytical calculations, visualization, writing the manuscript.

N.O. Afanasyeva—numerical calculations, investigations, visualization, and writing the manuscript.

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