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RESEARCH ARTICLE

Application of the Berreman formalism for modeling magneto-optical Kerr effects in multilayered structures

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Abstract
Objectives. Materials composed of numerous ultrathin layers, each having a thickness on the order of several nanometers, constitute an advanced class of composite structures exhibiting unique physical properties not typically found in conventional materials. These materials are of significant interest in both scientific and industrial sectors due to their adaptability and broad potential for application. Researchers are particularly intrigued by structures incorporating both magnetic and non-magnetic layers. The investigation of magneto-optical phenomena—particularly the Kerr effect—within these structures contributes to a deeper understanding of their physical characteristics, as well as enhancing prospects for their practical implementation. Since, to ensure the accurate interpretation of experimental data, it is imperative to consider potential interference effects, it becomes necessary to develop a mathematical model of the structure for comparing experimental findings with theoretical calculations. The purpose of this study is to analyze one of the modeling methods for multilayer structures in which magneto-optical Kerr effects can manifest themselves in individual or all layers.
Methods. The Berreman method, which is based on the matrix representation of Maxwell's differential equations, is used to model all three magneto-optical Kerr effects (polar, longitudinal, transverse) in multilayer thin-film structures.
Results. For optically isotropic materials, Berreman matrices have been derived for experimental configurations required to observe the transverse, polar, and longitudinal Kerr effects. A method is additionally proposed to account for the influence of thick layers within the investigated structure.
Conclusions. For the matrices presented in this paper, the Berreman method was used to analyze magneto-optical Kerr effects in an isotropic medium. As well as allowing us to obtain accurate formulas for magneto-optical effects, this provided more accurate modeling of complex multilayer structures, as well as contributing to an in-depth understanding of their physical characteristics, which provides new opportunities for analyzing and searching a wide range of materials.

Keywords: magneto-optical Kerr effects, Berreman method, Berreman matrix, dielectric constant tensor, multilayer structures

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НАУЧНАЯ СТАТЬЯ

Применение метода Берремана при моделировании магнитооптических эффектов Керра в многослойных структурах

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Резюме

Цели. Материалы, состоящие из множества ультратонких слоев, каждый из которых имеет толщину порядка нескольких нанометров, являются перспективным классом композитных структур с уникальными физическими характеристиками, не присущими традиционным материалам. Они представляют значительный интерес в научной и промышленной сферах благодаря своей многофункциональности и широким возможностям применения. Особое внимание исследователей привлекают структуры, включающие как магнитные, так и немагнитные слои. Исследование магнитооптических явлений, в частности эффекта Керра, в данных структурах способствует углублению понимания их физических свойств и расширению возможностей их практического применения. Для корректной интерпретации экспериментальных данных необходимо учитывать возможные интерференционные эффекты. В связи с этим возникает потребность в разработке математической модели структуры и сопоставлении экспериментальных результатов с теоретическими расчетами. Целью настоящего исследования является анализ одного из методов моделирования многослойных структур, в которых всесторонне теоретически рассматриваются все три магнитооптических эффекта Керра (полярный, меридиональный, экваториальный) с получением универсальных формул.

Методы. Для моделирования всех трех магнитооптических эффектов Керра в многослойных тонкопленочных структурах применяется метод Берремана, основанный на матричном представлении дифференциальных уравнений Максвелла.

Результаты. Для оптически изотропных материалов получены матрицы Берремана, соответствующие экспериментальным геометриям, необходимым для наблюдения экваториального, полярного и меридионального эффектов Керра. Предложен метод учета толстых слоев в исследуемой структуре.

Выводы. Использование метода Берремана с применением матриц, представленных в данной работе, для анализа магнитооптических эффектов Керра в изотропной среде позволило получить точные формулы магнитооптических эффектов и обеспечило более точное моделирование сложных многослойных структур, а также способствует углубленному пониманию их физических характеристик, открывая возможности для анализа и поиска широкого спектра материалов.

Ключевые слова: магнитооптические эффекты Керра, метод Берремана, матрица Берремана, тензор диэлектрической проницаемости, многослойные структуры

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INTRODUCTION

Michael Faraday's 1845 discovery of the interaction of light with magnetized matter subsequently led to the discovery of a multitude of linear and nonlinear, direct and inverse magneto-optical effects in the visible, infrared, ultraviolet, and X-ray regions of the spectrum. This field of physics is known as magneto-optics or magneto-photonics. Kerr effects, including magneto-optical reflection effects, provide the fundamental basis for magneto-optical Kerr spectroscopy method. This approach is used to analyze the magnetic state of local areas of a sample at the depth of formation of the magneto-optical signal. By studying the spectral dependencies of the magneto-optical Kerr effect at different wavelengths of radiation, it becomes possible to obtain information about magneto-optical transitions that reflect the electronic, crystalline, and magnetic structure of the local area of the sample under study [1–4]. Thus, magneto-optical spectroscopy is an important tool for studying micro- and nanostructures, including multilayer systems.

Multilayer magnetic structures are a subject of increased interest among researchers due to their significant potential for application in various fields of science and technology. The growing scientific interest in the study and development of these structures in recent years is accompanied by an increase in the number of scientific publications devoted to their research. Particular attention is paid to systems consisting of strongly magnetic (ferromagnets and ferrimagnets) and weakly magnetic materials [5–8].

However, modeling structures that include layers of materials with magneto-optical properties is a more complex task than calculating optical systems based on isotropic media. While, when obtaining analytical expressions for a certain number of magnetic layers, such as in [9], any increase in their number adds already represents a laborious task, with a significant number of magnetic layers, it becomes unfeasible. As a rule, the case of normal incidence of light on the structure is considered within the framework of the Jones matrix method.

BERREMAN METHOD

More than half a century ago, Berreman proposed a method for calculating the propagation of electromagnetic waves using 4 × 4 complex matrices based on the matrix representation of Maxwell's equations [10]. This highly accurate and versatile method is used to accounting for the peculiarities of light propagation in complex anisotropic structures, including multilayer systems. Despite its advantages, Berreman's method has not been widely used due to its high computational complexity and the difficulty of calculating matrices in general. Nevertheless, when solving problems related to optical anisotropy (uniaxial, biaxial) and the need to take into account magnetic anisotropy or optical activity of the medium, the advantages of the Berman method may turn out to be significant, thus justifying its application under the appropriate conditions.

When applying the Berman method, one of the most challenging problems arises when calculating the matrix exponent. However, a number of techniques have been developed to calculate this exponent with varying degrees of accuracy [11]. Moreover, if the accuracy is insufficient, the calculated layer can be divided into smaller sublayers. A fairly accurate and effective algorithm for calculating Berreman matrices for layers of considerable thickness can be found in the scientific literature [12]. In certain cases, such as a uniaxial medium, the Berreman matrix can be calculated analytically.

The present paper also provides accurate matrices for homogeneous media with induced optical activity for geometries corresponding to the transverse, polar, and longitudinal Kerr effects. The Berreman method is used to analyze the propagation of a plane monochromatic wave through a homogeneous medium. A one-dimensional inhomogeneous medium can be described by a system of plane-parallel layers, each of which can be considered homogeneous.

Let us consider the essence of the method using the coordinate system shown in Fig. 1. Let us also assume that there is air on both sides of the layer under investigation. Consequently, the medium from which light falls on the layer of material and into which it passes after passing through this layer is homogeneous, non-absorbing, and has a refractive index equal to one.

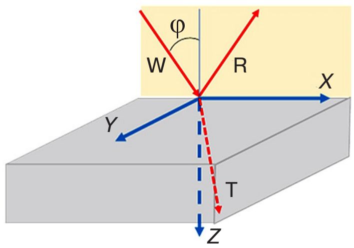


Fig. 1. Coordinate system used:
X, *Y*, *Z* are coordinate axes;
φ is the angle of incidence;
W is the incident (wave),
R is reflected, and
T is transmitted light beams

Since the wave is monochromatic, the time dependence of all components of the electric (E) and magnetic (H) fields has the following form: $e^{-i\omega t}$. Within the geometry under consideration, the projection of the wave vector onto the x-axis, denoted as k_x, is the same for all waves and takes the following value:

$$k_x = \frac{\omega}{c}\sin\varphi, \quad (1)$$

where ω is the frequency, φ is the angle of incidence, and c is the speed of light in a vacuum.

Then Maxwell's equations can be written as:

$$\mathbf{R}\Psi = -i\omega\mathbf{M}\Psi, \quad (2)$$

where **R, M** are 6 × 6 block matrices, while Ψ is a column matrix containing the following elements:

$$\mathbf{R} = \begin{pmatrix} \mathbf{O} & \mathbf{rot} \\ -\mathbf{rot} & \mathbf{O} \end{pmatrix}, \mathbf{M} = \begin{pmatrix} \hat{\varepsilon} & \mathbf{O} \\ \mathbf{O} & \bar{\mu} \end{pmatrix}, \Psi = \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix}.$$

Here **O** is a zero matrix of dimension 3 × 3 and, for our case,

$$\mathbf{rot} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} & 0 \\ \frac{\partial}{\partial z} & 0 & -ik_x \\ 0 & ik_x & 0 \end{pmatrix},$$

$$\hat{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix},$$

$$\hat{\mu} = \mu_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

ε_0 and μ_0 are the electric and magnetic constants, respectively.

In the process of solving Eq. (2), a system is formed that includes two linear homogeneous algebraic equations and four differential equations. The algebraic equations are solved with respect to the field components E_z and H_z, after which the resulting expressions are substituted into the differential equations. The result is a system of four linear homogeneous differential equations of the first order containing the unknown field components E_x, E_y, H_x, and H_y:

$$\frac{\partial}{\partial z}\xi = i\omega\Delta\xi, \quad (3)$$

where ξ is the matrix-column of the form:

$$\xi = \begin{pmatrix} E_x \\ H_y \\ E_y \\ -H_x \end{pmatrix},$$

and Δ is the differential propagation matrix for a given medium, with dimensions 4 × 4.

As mentioned earlier, the medium is divided into layers within which the components of the matrix Δ do not depend on the coordinate z. In each layer of thickness h the solution of the corresponding homogeneous first-order differential Eq. (3) has the form:

$$\xi(z + h) = e^{i\omega h\Delta}\,\xi(z) \equiv \mathbf{P}(h)\xi(z).$$

Thus, the **P** matrix, representing the Berman matrix of this layer, appears as follows:

$$\mathbf{P}(h) = e^{i\omega h\Delta}. \quad (4)$$

When the optical parameters of the medium depend on z, for example, for a structure consisting of homogeneous layers within which the optical parameters are considered constant, integrating Eq. (3) reduces to multiplying the corresponding matrices for individual layers:

$$\mathbf{P} = \prod_{j=1}^{n} \mathbf{P}_j.$$

The matrix of the next layer is multiplied by the previous ones on the left:

$$\xi(h) = \{\mathbf{P}_n \cdot \mathbf{P}_{n-1} \cdot \mathbf{P}_{n-2} \cdot \ldots \cdot \mathbf{P}_3 \cdot \mathbf{P}_2 \cdot \mathbf{P}_1\}\xi(0).$$

The electromagnetic field on one side of the structure is determined by the superposition of incident and reflected waves, while on the other side there is only the transmitted wave. Then:

$$\xi_T = \mathbf{P}(\xi_W + \xi_R). \quad (5)$$

According to the designations shown in Fig. 1, the indices W, R, and T denote incident, reflected, and transmitted waves, respectively. Multiplying both sides of Eq. (5) on the left by the matrix $\mathbf{F} = \mathbf{P}^{-1}$, representing the inverse of the Berman matrix of the layer, we obtain a system of linear equations that can be solved for the components of the reflected and transmitted waves. To identify the belonging to the corresponding wave, we assign upper indices to the field components corresponding to the designations in Fig. 1. The lower indices of the field components indicate the direction of the projection of the component under consideration onto the corresponding axis. Let us list them so that readers can apply them in their work.

$$E_y^{\mathrm{T}} = \frac{2\left(\chi_1 E_y^{\mathrm{W}} - \alpha\chi_2 E_x^{\mathrm{W}}\right)}{d - ag},\ E_x^{\mathrm{T}} = \frac{2\chi_2 E_x^{\mathrm{W}} - gE_y^{\mathrm{T}}}{b}, \quad (6)$$

$$E_y^{\mathrm{R}} = \beta_3 E_x^{\mathrm{T}} + \gamma_3 E_y^{\mathrm{T}} - E_y^{\mathrm{W}},\ E_x^{\mathrm{R}} = \beta_1 E_x^{\mathrm{T}} + \gamma_1 E_y^{\mathrm{T}} - E_x^{\mathrm{W}},$$

where

$$a = \frac{\beta_4 + \chi_1\beta_3}{\beta_2 + \chi_2\beta_1},\ b = \beta_2 + \chi_2\beta_1,\ g = \gamma_2 + \chi_2\gamma_1,\ d = \gamma_4 + \chi_1\gamma_3,$$

$$\beta_1 = f_{11} + \chi_2 f_{12},\ \gamma_1 = f_{13} + \chi_1 f_{14},$$

$$\beta_2 = f_{21} + \chi_2 f_{22},\ \gamma_2 = f_{23} + \chi_1 f_{24},$$

$$\beta_3 = f_{31} + \chi_2 f_{32},\ \gamma_3 = f_{33} + \chi_1 f_{34},$$

$$\beta_4 = f_{41} + \chi_2 f_{42},\ \gamma_4 = f_{43} + \chi_1 f_{44},$$

$$\chi_1 = \sqrt{\frac{\varepsilon_0}{\mu_0}}\cos\varphi,\ \chi_2 = \sqrt{\frac{\varepsilon_0}{\mu_0}}\cdot\frac{1}{\cos\varphi},$$

$f_{11}, f_{12}, \ldots, f_{44}$ are elements of the matrix $\mathbf{F} = \mathbf{P}^{-1}$, the inverse of the Berreman matrix.

Knowing the values of the light wave field components, it is possible to calculate the polarization rotation, as well as the reflection coefficient K_{R} and transmission coefficient K_{T}:

$$K_{\mathrm{R}} = \frac{|E_x^{\mathrm{R}}/\cos\varphi|^2 + |E_y^{\mathrm{R}}|^2}{|E_x^{\mathrm{W}}/\cos\varphi|^2 + |E_y^{\mathrm{W}}|^2},$$
$$K_{\mathrm{T}} = \frac{|E_x^{\mathrm{T}}/\cos\varphi|^2 + |E_y^{\mathrm{T}}|^2}{|E_x^{\mathrm{W}}/\cos\varphi|^2 + |E_y^{\mathrm{W}}|^2}, \quad (7)$$

but first it is necessary to determine the Berreman matrices for the modeled structure and its individual layers.

BERREMAN MATRICES FOR MODELING MAGNETO-OPTICAL KERR EFFECTS

Magneto-optical phenomena manifest themselves in changes in the optical properties of a film (structure) depending on the presence or absence of a magnetic field. In this regard, their modeling requires determining the Berreman matrices for both scenarios. In the cases considered here, they can all be obtained analytically.

The method for finding the matrix for an isotropic layer in the absence of a magnetic field was demonstrated by Berreman in one of his early works [10]. Substituting the dielectric permeability tensor in the form:

$$\hat{\varepsilon} = \varepsilon_0\varepsilon\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we obtain a differential, 4 × 4 dimensional, propagation matrix Δ_0 for a given medium:

$$\Delta_0 = \begin{pmatrix} 0 & U_a & 0 & 0 \\ U_b & 0 & 0 & 0 \\ 0 & 0 & 0 & U_s \\ 0 & 0 & U_d & 0 \end{pmatrix}, \quad (8)$$

where

$$U_a = \frac{\mu_0}{\varepsilon}(\varepsilon - \sin^2\varphi),\ U_b = \varepsilon_0\varepsilon,\ U_s = \mu_0,\ U_d = \varepsilon_0(\varepsilon - \sin^2\varphi). \quad (9)$$

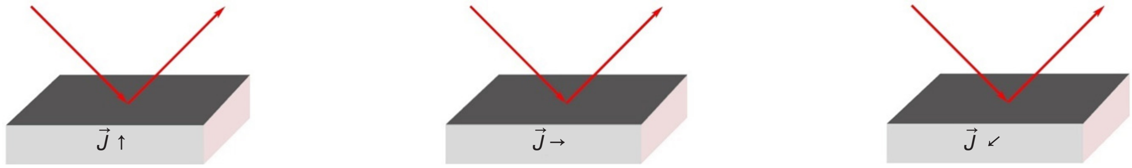
According to Eq. (4), to obtain the Berman matrix, it is necessary to take the exponential of matrix (8) multiplied by $i\omega h$. The simplest and most intuitive way to do this is to expand the exponential into a Taylor series. Summing the terms of the series, we can see that the elements of the resulting matrix also represent Taylor series expansions of certain functions. As a result, for the Berman matrix of an isotropic homogeneous medium $\mathbf{P_0}$, we can write the following:

$$\mathbf{P_0}(h) = \begin{pmatrix} \cos(\omega h\chi_0) & i\beta_0\sin(\omega h\chi_0) & 0 & 0 \\ i\beta_0^{-1}\sin(\omega h\chi_0) & \cos(\omega h\chi_0) & 0 & 0 \\ 0 & 0 & \cos(\omega h\chi_0) & i\delta_0^{-1}\sin(\omega h\chi_0) \\ 0 & 0 & i\delta_0\sin(\omega h\chi_0) & \cos(\omega h\chi_0) \end{pmatrix}. \quad (10)$$

Here

$$\chi_0 = \frac{1}{c}\sqrt{\varepsilon - \sin^2\varphi},\ \beta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot\frac{\sqrt{\varepsilon - \sin^2\varphi}}{\varepsilon},\ \delta_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon - \sin^2\varphi}. \quad (11)$$

Depending on the configuration of the magnetic field and the incidence of light on the film or structure, three magneto-optical Kerr effects are distinguished: polar, longitudinal, and transverse. The polar Kerr effect occurs when the magnetization vector J, created in particular by an external magnetic field, is oriented perpendicular to the plane of the film and parallel to the plane of incidence of light (Fig. 2a). The longitudinal Kerr effect occurs when the magnetization vector is oriented parallel to the film surface and lies in the plane of light incidence (Fig. 2b). Conversely, the transverse Kerr effect is observed when the magnetization vector is oriented perpendicular to the plane of incidence of light and parallel to the film structure (Fig. 2c).



(a) (b) (c)

Fig. 2. Geometry of observation of magneto-optical Kerr effects: (a) polar, (b) longitudinal, (c) transverse. The directions of the coordinate axes are shown in Fig. 1

It should be noted that the s- and p-components of the electric field of a light wave in this geometry are related to the projections on the axes as follows: $E_s = E_y$, $E_p = E_x/\cos\varphi$. Knowing E_s and E_p, we can calculate the angle of rotation of the polarization plane for a plane-polarized wave $\theta_0 = \mathrm{arctg}(|E_s|/|E_p|)$ or, if there is a non-zero phase difference ϕ between E_s and E_p, the azimuth angle θ_ϕ, which forms the main axis of the polarization ellipse with the plane of incidence $\theta_\phi = \mathrm{arctg}(|E_s|/|E_p|)\cos\phi$. The ellipticity e can be calculated using the formula: $e = \mathrm{tg}[0.5\arcsin(-2\mathrm{Im}(\Xi)/(1-|\Xi|^2))]$, where $\Xi = (|E_s|/|E_p|)e^{i\phi}$ [13, 14]. These are precisely the values measured in the longitudinal and polar Kerr effects, while in the transverse effect, the main factor is the change in the intensity of the reflected electromagnetic wave.

Let us begin our consideration with the transverse Kerr effect (Fig. 2c). In this case, the dielectric permeability tensor of the medium in the chosen coordinate system can be written as:

$$\hat{\varepsilon} = \varepsilon_0\varepsilon\begin{pmatrix} 1 & 0 & iQ \\ 0 & 1 & 0 \\ -iQ & 0 & 1 \end{pmatrix},$$

where Q is the magneto-optical parameter.

In this case, solving Eq. (2) yields the following propagation matrix Δ_T (T is transverse Kerr effect):

$$\Delta_\mathrm{T} = \begin{pmatrix} i\eta & U_a & 0 & 0 \\ U_b^* & -i\eta & 0 & 0 \\ 0 & 0 & 0 & U_s \\ 0 & 0 & U_d & 0 \end{pmatrix}. \quad (12)$$

Here

$$U_b^* = U_b(1-Q^2),\ \ \eta = \frac{\sin\varphi}{\mathrm{c}}Q.$$

The matrix (12) can be represented as the sum of matrices:

$$\Delta_\mathrm{T} = \Delta_\mathrm{T}^J + \Delta_\mathrm{T}^* = \begin{pmatrix} i\eta & 0 & 0 & 0 \\ 0 & -i\eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & U_a & 0 & 0 \\ U_b^* & 0 & 0 & 0 \\ 0 & 0 & 0 & U_s \\ 0 & 0 & U_d & 0 \end{pmatrix}.$$

Since the structure of the matrix Δ_T^* is identical to the matrix structure Δ_0, the matrix Δ_T^* will have a form similar to (10). We will also find the exponent of the matrix Δ_T^J, multiplied by $i\omega h$, using Taylor series expansion. Fortunately, in this case, the elements of the resulting matrix also represent series expansions. As a result, we obtain:

$$e^{i\omega h\Delta_\mathrm{T}^J} = \begin{pmatrix} e^{-\omega h\eta} & 0 & 0 & 0 \\ 0 & e^{\omega h\eta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, the Berreman matrix in this case will have the form:

$$\mathbf{P}_{\mathrm{T}}(h) = \mathrm{e}^{i\omega h\Delta_{\mathrm{T}}^{J}}\mathrm{e}^{i\omega h\Delta_{\mathrm{T}}^{*}}, \tag{13}$$

$$\mathbf{P}_{\mathrm{T}}(h) = \begin{pmatrix} \mathrm{e}^{-\omega h\eta}\cos(\omega h\chi_{\mathrm{T}}) & i\,\mathrm{e}^{-\omega h\eta}\,\beta_{\mathrm{T}}\sin(\omega h\chi_{\mathrm{T}}) & 0 & 0 \\ i\,\mathrm{e}^{\omega h\eta}\,\beta_{\mathrm{T}}^{-1}\sin(\omega h\chi_{\mathrm{T}}) & \mathrm{e}^{\omega h\eta}\cos(\omega h\chi_{\mathrm{T}}) & 0 & 0 \\ 0 & 0 & \cos(\omega h\chi_0) & i\delta_0^{-1}\sin(\omega h\chi_0) \\ 0 & 0 & i\delta_0\sin(\omega h\chi_0) & \cos(\omega h\chi_0) \end{pmatrix}, \tag{14}$$

where

$$\chi_{\mathrm{T}} = \chi_0\sqrt{1-Q^2},\ \beta_{\mathrm{T}} = \beta_0\Big/\sqrt{1-Q^2}.$$

In the process of calculating matrix (14), a transition was made from the operation of adding matrices to the operation of multiplying their exponents, which are also matrices. It should be noted that, unlike addition, matrix multiplication is generally not commutative. If the reverse order of multiplication is used, matrix (14) will have a different form: the changes will affect elements P_{12} and P_{21}, resulting in the exponents included in them being swapped.

The multiplication order shown in (13) was chosen for the following reasons. First, based on general considerations, modifications caused by the magneto-optical effect manifest themselves after light radiation passes through the film (structure), rather than before it interacts with it. Second, the results of calculations [15] of the transverse Kerr effect for a cobalt film on a silicon substrate (Fig. 3) demonstrate that using the multiplication order (13) yields a result that is consistent with the experiments in terms of the nature of the spectral dependence and the order of magnitude. At the same time, the use of the reverse multiplication order leads to simulation results that do not fully correspond to the experimental data.

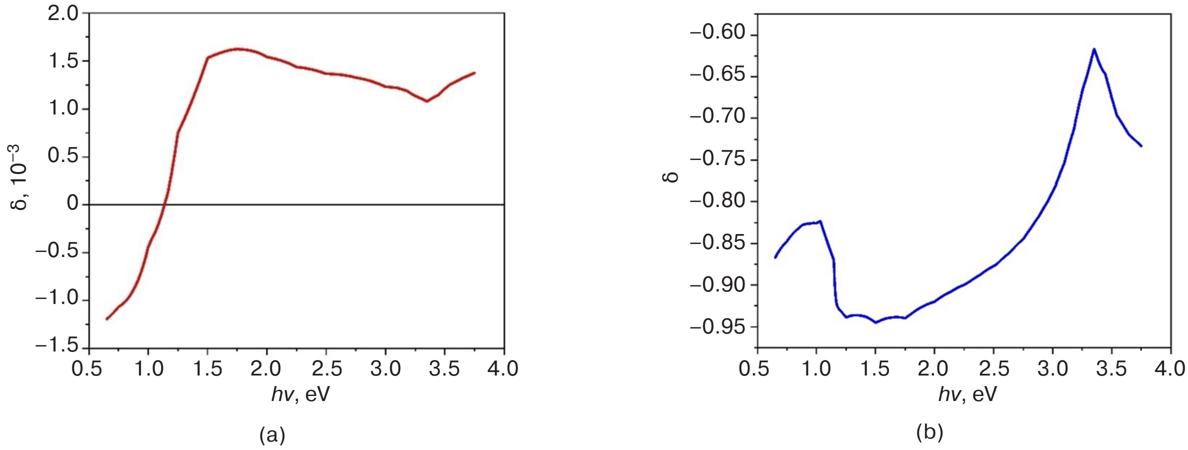


Fig. 3. Results of modeling the spectral dependence of the transverse Kerr effect (δ) of a cobalt film on a silicon substrate, taking into account the order of matrix multiplication: (a) presented in (13), (b) reverse. *hν* is photon energy

For the case of the polar Kerr effect (Fig. 2a), the permittivity tensor in the coordinates used can be represented as

$$\hat{\varepsilon} = \varepsilon_0\varepsilon\begin{pmatrix} 1 & iQ & 0 \\ -iQ & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From (2) we obtain:

$$\Delta_{\mathrm{P}} = \begin{pmatrix} 0 & U_a & 0 & 0 \\ U_b & 0 & i\zeta & 0 \\ 0 & 0 & 0 & U_s \\ -i\zeta & 0 & U_d & 0 \end{pmatrix}, \tag{15}$$

where $\zeta = \varepsilon_0\varepsilon Q$.

Here we can also consider the sum of the matrix (7) and

$$\Delta_{\mathrm{P}}^{J}=\begin{pmatrix}0&0&0&0\\0&0&i\zeta&0\\0&0&0&0\\-i\zeta&0&0&0\end{pmatrix}. \tag{16}$$

Due to the fact that the second degree of the matrix (16) is already equal to zero,

$$e^{i\omega h\Delta_{\mathrm{P}}^{J}}=\begin{pmatrix}1&0&0&0\\0&1&-\omega h\zeta&0\\0&0&1&0\\\omega h\zeta&0&0&1\end{pmatrix}.$$

The order of matrix multiplication is considered when deriving the expression for the transverse effect. As a result, the Berreman matrix for the polar effect takes the form:

$$\mathbf{P}_{\mathrm{P}}(h)=\begin{pmatrix}\cos(\omega h\chi_0)&i\beta_0\sin(\omega h\chi_0)&0&0\\i\beta_0^{-1}\sin(\omega h\chi_0)&\cos(\omega h\chi_0)&-\omega h\zeta\cos(\omega h\chi_0)&-i\omega h\zeta\delta_0^{-1}\sin(\omega h\chi_0)\\0&0&\cos(\omega h\chi_0)&i\delta_0^{-1}\sin(\omega h\chi_0)\\\omega h\zeta\cos(\omega h\chi_0)&i\omega h\zeta\beta_0\sin(\omega h\chi_0)&i\delta_0\sin(\omega h\chi_0)&\cos(\omega h\chi_0)\end{pmatrix}. \tag{17}$$

Using similar reasoning, we can derive the Berreman matrix for the case of the longitudinal Kerr effect (Fig. 2b). In this case:

$$\hat{\varepsilon}=\varepsilon_0\varepsilon\begin{pmatrix}1&0&0\\0&1&iQ\\0&-iQ&1\end{pmatrix}$$

and

$$\mathbf{P}_{\mathrm{M}}(h)=\begin{pmatrix}\cos(\omega h\chi_0)&i\beta_0\sin(\omega h\chi_0)&-\rho\cos(\omega h\chi_{\mathrm{M}})&-i\rho\delta_{\mathrm{M}}^{-1}\sin(\omega h\chi_{\mathrm{M}})\\i\beta_0^{-1}\sin(\omega h\chi_0)&\cos(\omega h\chi_0)&0&0\\0&0&\cos(\omega h\chi_{\mathrm{M}})&i\delta_{\mathrm{M}}^{-1}\sin(\omega h\chi_{\mathrm{M}})\\i\rho\beta_0^{-1}\sin(\omega h\chi_0)&\rho\cos(\omega h\chi_0)&i\delta_{\mathrm{M}}\sin(\omega h\chi_{\mathrm{M}})&\cos(\omega h\chi_{\mathrm{M}})\end{pmatrix}. \tag{18}$$

Here

$$\rho=\frac{\omega}{c}Q\sin\varphi,\ \chi_{\mathrm{M}}=\frac{1}{c}\sqrt{\varepsilon(1+Q^2)-\sin^2\varphi},\ \delta_{\mathrm{M}}=\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\varepsilon(1+Q^2)-\sin^2\varphi}.$$

THICK LAYERS CONSIDERATION

The Berreman method takes into account interference effects. However, the simulated structures may contain fairly thick layers, the thickness of which exceeds the coherence length l_{coh}. of the light source. While such layers typically represent substrates onto which films and complex structures are deposited, the structures under study may also include a certain number of thick layers. The principle of taking such layers into account is based on the averaging method, in which deviations from coherence are considered as random variables obeying a normal distribution [16]. Although this approach requires additional computing power, it enables the inclusion of layers of intermediate thickness, in which a partial violation of the coherence condition is observed. In the elements of the Berreman matrix of a thick layer, the frequency ω is replaced by the sum (ω + w), where w is a random variable distributed according to the normal law. All frequency-dependent characteristics of the medium remain unchanged. Thus, the introduction of the variable w affects only the phase change. The mathematical expectation M_w = 0. The standard deviation σ_w depends on the layer thickness and is taken from the condition that with a layer thickness equal to the coherence length of the radiation, a phase shift π will occur:

$$\sigma_w=\frac{h}{l_{\mathrm{coh}}}\pi.$$

The magnitude of the calculated effect for the entire structure is averaged. Consider the case where a thin-film structure is deposited on a thick substrate with a thickness h_{sub}. Then, the reflectance from such a structure is calculated as:

$$\langle R \rangle = \frac{1}{\sqrt{2\pi^3}} \cdot \frac{l_{\text{coh}}}{h_{\text{sub}}} \int_{-\infty}^{\infty} R'(w) e^{-\frac{l_{\text{coh}}^2}{2\pi 2 h_{\text{sub}}^2} w^2} dw.$$

The values of $R'(w)$ are determined in accordance with Eq. (7) for the radiation frequency ω_0. In this case, in the matrices of all layers, except for the layer being averaged, $\omega = \omega_0$ is taken. In the averaged layer $\omega = \omega_0 + w$ is taken. All material characteristics for all layers are taken for the frequency ω_0. If there are several thick layers, multiple integration is performed.

Another problem with thick layers is that, with high absorption, an overflow situation may occur during intermediate calculations for the matrix or its inverse. To address this issue, it is proposed to introduce a certain error into the calculations. Accordingly, the maximum allowable thickness of the absorbing layers will be determined based on the condition that, when passing through these layers, the light wave amplitude decreases by 5–7 orders of magnitude. This will avoid data overflow, and the introduced error will be clearly smaller than the experimental errors.

CONCLUSIONS

This study proposes using the Berreman method to model all three magneto-optical Kerr effects: polar, longitudinal, and transverse. The Berreman matrix method presented in this paper enables a systematic analysis of magneto-optical Kerr effects in an isotropic medium using the obtained precise formulas for magneto-optical effects. As well as providing a sound basis for more accurate modeling of complex multilayer structures, this facilitates a deeper understanding of their physical characteristics, opening up possibilities for exploring a wide range of promising magnetic materials. The approaches for accounting for the presence of films in the modeled structure whose thickness exceeds the coherence length of the light source in this paper expand the range of micro- and nanostructures under consideration.

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Authors' contributions

I.V. Gladyshev—methodology development, modeling, discussion of the results, drafting an article and its editing.

A.N. Yurasov, M.M. Yashin—discussion of the results, preparation of the article and its editing.

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