

Mathematical modeling
Математическое моделирование

UDC 531.32

<https://doi.org/10.32362/2500-316X-2026-14-1-64-81>

EDN JUUJON



RESEARCH ARTICLE

Mathematical modeling of the orbital motion of an artificial satellite of the Moon using Delaunay variables

Olga V. Meshkova, Albina V. Shatina [®]*MIREA – Russian Technological University, Moscow, 119454 Russia*[®] *Corresponding author, e-mail: shatina_av@mail.ru*

• Submitted: 19.05.2025 • Revised: 10.07.2025 • Accepted: 07.11.2025

Abstract

Objectives. This work aims to derive and study the system of equations of orbital motion of an artificial satellite of the Moon (ASM) in the gravitational field of an attracting planet using Delaunay variables. This will ensure a reduction in computational complexity when modeling long-term trajectories, as well as provide an analysis of stationary orbits of the Moon taking into account the gravitational influence of the Earth as a third body.

Methods. The study uses analytical mechanics, asymptotic methods, in particular, the averaging method, methods of stability theory, numerical methods for integrating systems of ordinary differential equations.

Results. The Hamiltonian and equations of motion of the ASM in canonical Delaunay variables are obtained. Averaged and non-averaged systems of equations of motion of the ASM are derived in the form of autonomous systems of ordinary differential equations with respect to the following orbital parameters: semi-major axis, eccentricity, inclination, longitude of the ascending node, longitude of the pericenter from the ascending node, and true anomalies. A closed system of differential equations of the second order with respect to the orbital eccentricity and the pericenter longitude from the ascending node is obtained. Its stationary solutions are found, their stability is investigated, and conditions for the existence of stationary motions are determined depending on the value of the constant of the first integral of the averaged system of equations. Integral curves and phase portraits were constructed to demonstrate the interrelationship of orbital parameters. A comparative analysis was conducted using JPL Horizons¹ data and previously published works.

Conclusions. The method developed enables the design of trajectories for future lunar missions to be optimized (e.g., Artemis, Luna-Glob), thus providing a balance between accuracy and computational efficiency. The results confirm the prospects of using Delaunay variables for analyzing long-term orbital dynamics in gravitational fields of complex configuration.

Keywords: artificial satellite of the Moon, gravitational field of the attracting center, Hamiltonian, canonical Delaunay variables, system of equations of orbital motion, orbital parameters

¹ JPL Horizons is an online service from the National Aeronautics and Space Administration (USA) that provides access to key data about the solar system and enables the calculation of precise trajectories of objects in it. <https://ssd.jpl.nasa.gov/horizons/>. Accessed March 04, 2025.

For citation: Meshkova O.V., Shatina A.V. Mathematical modeling of the orbital motion of an artificial satellite of the Moon using Delaunay variables. *Russian Technological Journal*. 2026;14(1):64–81. <https://doi.org/10.32362/2500-316X-2026-14-1-64-81>, <https://www.elibrary.ru/JUUJON>

Financial disclosure: The authors have no financial or proprietary interest in any material or method mentioned.

The authors declare no conflicts of interest.

НАУЧНАЯ СТАТЬЯ

Математическое моделирование орбитального движения искусственного спутника Луны с использованием переменных Делоне

О.В. Мешкова, А.В. Шатина[@]

МИРЭА – Российский технологический университет, Москва, 119454 Россия

[@] Автор для переписки, e-mail: shatina_av@mail.ru

• Поступила: 19.05.2025 • Доработана: 10.07.2025 • Принята к опубликованию: 07.11.2025

Резюме

Цели. Целью работы является вывод и исследование системы уравнений орбитального движения искусственного спутника Луны (ИСЛ) в гравитационном поле притягивающей планеты в переменных Делоне, обеспечивающей снижение вычислительной сложности при моделировании долгосрочных траекторий, а также анализ стационарных орбит Луны с учетом гравитационного влияния Земли как третьего тела.

Методы. Используются методы аналитической механики, асимптотические методы, в частности, метод усреднения, методы теории устойчивости, численные методы для интегрирования систем обыкновенных дифференциальных уравнений.

Результаты. Получены гамильтониан и уравнения движения ИСЛ в канонических переменных Делоне, на основе которых выведены усредненная и неусредненная системы уравнений движения ИСЛ в виде автономных систем обыкновенных дифференциальных уравнений относительно следующих параметров орбиты: большой полуоси, эксцентриситета, наклона, долготы восходящего узла, долготы перицентра от восходящего узла, истинных аномалий. Получена замкнутая система дифференциальных уравнений второго порядка относительно эксцентриситета орбиты и долготы перицентра от восходящего узла. Найдены ее стационарные решения, исследована их устойчивость, определены условия для существования стационарных движений в зависимости от значения константы первого интеграла усредненной системы уравнений. Построены интегральные кривые и фазовые портреты, демонстрирующие взаимосвязь параметров орбиты. Проведен сравнительный анализ с данными JPL Horizons² и ранее опубликованными работами.

Выводы. Разработанный метод позволяет оптимизировать проектирование траекторий для будущих лунных миссий (например, Artemis, «Луна-Глоб»), обеспечивая баланс между точностью и вычислительной эффективностью. Результаты подтверждают перспективность использования переменных Делоне для анализа долгосрочной орбитальной динамики в гравитационных полях сложной конфигурации.

Ключевые слова: искусственный спутник Луны, гравитационное поле притягивающего центра, гамильтониан, канонические переменные Делоне, система уравнений орбитального движения, параметры орбиты

² JPL Horizons – онлайн-сервис Национального управления по аэронавтике и исследованию космического пространства (США), который предоставляет доступ к ключевым данным о Солнечной системе и позволяет вычислять точные траектории объектов в ней. <https://ssd.jpl.nasa.gov/horizons/>. Дата обращения 04.03.2025. [JPL Horizons is an online service from the National Aeronautics and Space Administration (USA) that provides access to key data about the solar system and enables the calculation of precise trajectories of objects in it. <https://ssd.jpl.nasa.gov/horizons/>. Accessed March 04, 2025.]

Для цитирования: Мешкова О.В., Шатина А.В. Математическое моделирование орбитального движения искусственного спутника Луны с использованием переменных Делоне. *Russian Technological Journal*. 2026; 14(1):64–81. <https://doi.org/10.32362/2500-316X-2026-14-1-64-81>, <https://www.elibrary.ru/JUUJON>

Прозрачность финансовой деятельности: Авторы не имеют финансовой заинтересованности в представленных материалах или методах.

Авторы заявляют об отсутствии конфликта интересов.

INTRODUCTION

The mathematical modeling of the orbital motion of an artificial satellite of the Moon (ASM) is one of the most pressing problems in space research. The relevance of this work is enhanced by the growing interest in long-term lunar projects, such as: the Artemis program (Artemis National Aeronautics and Space Administration, USA, 2022) [1]; the Chang'E-5 and Chang'E-6 missions (China National Space Administration, China, 2020 and 2024) [2, 3]; Chandrayaan-3 (Indian Space Research Organization, India, 2023) [4, 5]; and the Russian lunar program [6] which includes the deployment of an orbital station. In the context of the intensification of the lunar program in Russia, including the launch of new missions and research projects, accurate models of satellite motion are becoming especially necessary. The Moon is considered not only as an object of scientific study, but also as a potential base for further space research [6].

Although orbital motion around the Moon obeys Kepler's laws, significant deviations from the idealized model are explained by the influence of external disturbances. The fundamentals of analyzing such disturbances are laid out in the works of Laplace and Lagrange who proposed methods for solving the equations of motion for many-body systems³.

In the 1960s, studies of the evolution of planetary satellite orbits with a twice-averaged perturbation function were carried out. A detailed study of this problem began with the discovery of a new first integral of the averaged system of equations. This discovery was made almost simultaneously in 1961 by the Soviet scientist M.L. Lidov [7] and the American scientist Y. Kozai [8] and was called the Lidov–Kozai effect. Subsequently, a picture of the evolution of a satellite's orbital motion, based on this effect, was developed in the works of M.A. Vashkovyak [9, 10]. Lidov was the first to conduct a thorough study of the influence of gravitational perturbations on the motion of satellites in systems with several bodies. In his works [7, 11], he considered the effects caused by the gravity of a third body which lead to long-period changes in orbital parameters. In addition, he

proposed methods for simplifying complex equations of motion for practical application.

In turn, T.A. Eli [12] focused on numerical modeling of the orbital motion of lunar satellites taking into account significant external disturbances. In studies published in the early 2000s, he considered both short-term and long-term changes in orbital parameters under the influence of the gravity of the Earth, the Sun, and the Moon's mascons. Eli T.A. showed that for satellites in low lunar orbits (Low Lunar Orbit, LLO), the main influence is exerted by gravitational anomalies of the Moon, whereas for satellites in high lunar orbits (High Lunar Orbit, HLO), disturbances from the Earth and the Sun dominate.

In recent years, significant progress has been made in modeling lunar gravity anomalies. Study [13] demonstrated the use of high-order spherical harmonics to account for mascons, critical for low lunar orbits. Work [14] is devoted to the analysis of orbital stability within the framework of the Artemis mission, in which the authors use hybrid methods (analytical and numerical) to predict long-term evolution. In 2022, machine learning algorithms were proposed in [15], in the aims of accelerating the calculations of orbital perturbations especially relevant for real-time problems. In addition, the Chandrayaan-3 mission [4, 5] provided new data on the dynamics of highly elliptical orbits, confirming theoretical models. Work [16] proposed an approach to the design of frozen low lunar orbits based on the sequential application of non-gradient methods, Bayesian optimization and the Nelder–Mead method.

The motion of a lunar satellite is subject to various perturbations, including the gravitational influences of the Earth, the Sun, and other celestial bodies, the non-sphericity of the Moon, the influence of mascons (mass concentrations) in its lithosphere, solar radiation pressure, interactions with the space environment, and others. This article will focus on the system of equations for the spatial perturbation of the ASM motion caused by the Earth's gravitational field. This is the primary perturbation for satellites in high orbits.

Modern trajectory calculation systems rely on highly complex numerical methods which require significant resources. The proposed model, based on satellite orbital motion equations in Delaunay variables, provides comparable accuracy while reducing computation time. This makes the method promising for preliminary orbital analysis during the mission design phase.

³ Lagrange J.-L. *Mécanique Analytique*. Paris: Veuve Desaint, 1788.

1. TASK STATEMENT. HAMILTONIAN

Let us consider a model problem of the motion of a celestial body–satellite system in the gravitational field of an attracting center. The celestial body, satellite, and attracting center will be modeled by material points P , S , O with the masses m_2 , m_3 , m_1 , respectively (Fig. 1). We will assume that $m_3 \ll m_2 \ll m_1$. The radius vector of the center of mass C_0 of the celestial body–satellite system is denoted by the vector \mathbf{R}_1 , and the relative position of points P and S is denoted by the vector \mathbf{R}_2 , i.e., $\mathbf{R}_1 = \overline{OC_0}$, $\mathbf{R}_2 = \overline{PS}$. We will assume that $|\mathbf{R}_1| \gg |\mathbf{R}_2|$.

Let us introduce the inertial coordinate system $OXYZ$ and the Koenig axes $C_0\xi_1\xi_2\xi_3$. The radius vectors of the points P and S are expressed through \mathbf{R}_1 , \mathbf{R}_2 as follows:

$$\overline{OP} = \mathbf{R}_1 - \frac{m_3}{m_2 + m_3} \mathbf{R}_2, \quad \overline{OS} = \mathbf{R}_1 + \frac{m_2}{m_2 + m_3} \mathbf{R}_2. \quad (1.1)$$

The kinetic energy of the celestial body–satellite system is determined by the equity:

$$T = \frac{1}{2} m_2 \left(\frac{d}{dt} \overline{OP} \right)^2 + \frac{1}{2} m_3 \left(\frac{d}{dt} \overline{OS} \right)^2.$$

Taking into account relations (1.1), we obtain:

$$T = \frac{1}{2} (m_2 + m_3) \dot{\mathbf{R}}_1^2 + \frac{1}{2} m_r \dot{\mathbf{R}}_2^2, \quad m_r = \frac{m_2 m_3}{m_2 + m_3}, \quad (1.2)$$

m_r is the attached mass.

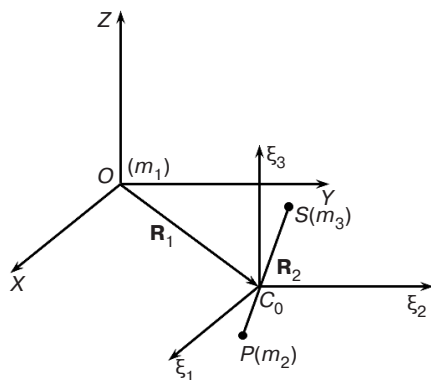


Fig. 1. Celestial body–satellite system

The potential energy of mutual attraction forces in the given problem is represented as:

$$\Pi = - \frac{f m_1 m_2}{\left| \mathbf{R}_1 - \frac{m_3}{m_2 + m_3} \mathbf{R}_2 \right|} - \frac{f m_1 m_3}{\left| \mathbf{R}_1 + \frac{m_2}{m_2 + m_3} \mathbf{R}_2 \right|} - \frac{f m_2 m_3}{|\mathbf{R}_2|}, \quad (1.3)$$

where $f = 6.672 \cdot 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}$ is the universal gravitational constant.

Given that the distance between the celestial body and the satellite is small compared to the length of the radius vector of the center of mass C_0 , and using the formulas for the generating function of Legendre polynomials $P_n(x)$ [17], while retaining the second-order terms of smallness in R_2/R_1 , we can transform expression (1.3). The potential (1.3) will take the following form:

$$\Pi = - \frac{f m_1 (m_2 + m_3)}{R_1} - \frac{f m_2 m_3}{R_2} - \frac{f m_1 m_r}{2 R_1} (3 \cos^2 \psi_{12} - 1) \frac{R_2^2}{R_1^2}. \quad (1.4)$$

Here $R_1 = |\mathbf{R}_1|$, $R_2 = |\mathbf{R}_2|$, ψ_{12} is the angle between vectors \mathbf{R}_1 , \mathbf{R}_2 .

As a basic task, let us consider the motion of satellite S_0 with mass m (material point) in the gravitational field of the attracting center Q . In this case, the satellite moves along a Keplerian orbit—one of the conic sections. Let us focus on the case of an elliptical orbit.

The following parameters are used to determine the position of a satellite in the orbit: a is the semi-major axis of the ellipse; e is the eccentricity; i is the inclination of the satellite's orbit; h is the longitude of the ascending node (angle between Qx and the line QN_1 of the intersection of the plane of the satellite's orbit with the plane Qxy); g is the longitude of the pericenter π from the ascending node (the point of the satellite's orbit closest to the center Q); ϑ is the true anomaly, the angle between the direction to the pericenter π and the vector \mathbf{R} , $\mathbf{G} = \mathbf{R} \times m \dot{\mathbf{R}}$ is the momentum vector of the point S_0 (Fig. 2).

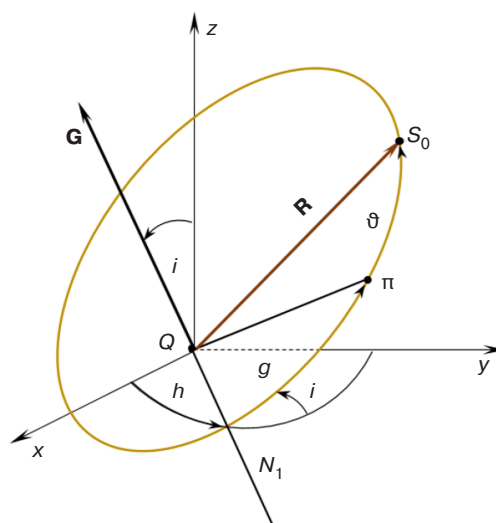


Fig. 2. Satellite orbit elements

The values h, i, g, e, n, a in the undisturbed problem remain constant, and the true anomaly ϑ is a function of time:

$$\dot{\vartheta} = \frac{(1 + e \cos \vartheta)^2}{(1 - e^2)^{3/2}} n, \quad n = \sqrt{\frac{f_0}{a^3}},$$

$f_0 = fM_0$, where M_0 is the mass of the center of attraction.

Kinetic and potential energies in this case are expressed as:

$$T = \frac{1}{2} m \dot{\mathbf{R}}^2, \quad \Pi = -\frac{f_0 m}{R}. \quad (1.5)$$

In celestial mechanics, canonical Delaunay variables L, G, H, l, g, h are used to describe the orbital motion of celestial bodies. These variables are related to the modulus R of the satellite's radius vector and elements a, e, i by the following equations [18–20]:

$$\begin{aligned} L &= m\sqrt{f_0 a}, \quad G = m\sqrt{f_0 a(1 - e^2)}, \\ H &= m\sqrt{f_0 a(1 - e^2)} \cos i, \\ R &= \frac{G^2}{f_0 m^2 (1 + e \cos \vartheta)}. \end{aligned}$$

Here g, h, i, ϑ are the previously defined orbital parameters, l is the mean anomaly. The dependence of the true anomaly ϑ on the variables l, L, G is implicitly expressed using the eccentric anomaly w through the following relationships:

$$\cos w = \frac{e + \cos \vartheta}{1 + e \cos \vartheta}, \quad l = w - e \sin w.$$

The Hamiltonian of the unperturbed problem with kinetic and potential energies (1.5) in Delaunay variables has a simple form:

$$\mathcal{H} = -\frac{f_0^2 m^3}{2L^2}.$$

Let us return to the task at hand with kinetic energy (1.2) and potential energy (1.4). We will describe the motion of the mechanical system using canonical variables Delaunay $L_k, G_k, H_k, l_k, g_k, h_k$, corresponding to the vector \mathbf{R}_k ($k = 1, 2$). The Hamiltonian of the task, taking this into account, will take the form:

$$\begin{aligned} \mathcal{H} &= -\frac{(fm_1)^2 (m_2 + m_3)^3}{2L_1^2} - \\ &- \frac{f^2 (m_2 + m_3)^2 m_r^3}{2L_2^2} + \mathcal{H}_1, \end{aligned} \quad (1.6)$$

$$\mathcal{H}_1 = -\frac{1}{2} \frac{fm_1 m_r}{R_1} (3 \cos^2 \psi_{12} - 1) \frac{R_2^2}{R_1^2}. \quad (1.7)$$

In formula (1.7), the modules of the vectors $\mathbf{R}_1, \mathbf{R}_2$ are expressed in terms of Delaunay variables as follows ($k = 1, 2$):

$$\begin{aligned} R_1 &= \frac{G_1^2}{fm_1 (m_2 + m_3)^2 (1 + e_1 \cos \vartheta_1)}, \\ R_2 &= \frac{G_2^2}{f(m_2 + m_3) m_r^2 (1 + e_2 \cos \vartheta_2)}, \end{aligned} \quad (1.8)$$

$$e_k = \sqrt{1 - \frac{G_k^2}{L_k^2}}.$$

At the same time, there are equalities:

$$\cos w_k = \frac{e_k + \cos \vartheta_k}{1 + e_k \cos \vartheta_k}, \quad l_k = w_k - e_k \sin w_k, \quad k = 1, 2.$$

For large orbital semi-axes and cosine of the inclination angle, expressions using Delaunay variables have the following form:

$$\begin{aligned} a_1 &= \frac{L_1^2}{fm_1 (m_2 + m_3)^2}, \quad a_2 = \frac{L_2^2}{f(m_2 + m_3) m_r^2}, \\ \cos i_2 &= \frac{H_2}{G_2}. \end{aligned} \quad (1.9)$$

Due to the conditions of the problem, we can assume that points P and S do not affect the motion of the center of mass C_0 , i.e., point C_0 moves along an undisturbed Keplerian elliptical orbit in the OXY plane. Let us write out vectors $\mathbf{R}_1, \mathbf{R}_2$ in the inertial coordinate system $OXYZ$.

$$\mathbf{R}_1 = R_1 \boldsymbol{\eta}_1, \quad \boldsymbol{\eta}_1 = (\cos(g_1 + \vartheta_1), \sin(g_1 + \vartheta_1), 0),$$

$$\begin{aligned} R_1 &= \frac{a_1 (1 - e_1^2)}{1 + e_1 \cos \vartheta_1}, \quad \dot{\vartheta}_1 = \frac{(1 + e_1 \cos \vartheta_1)^2}{(1 - e_1^2)^{3/2}} n_1, \\ n_1 &= \sqrt{\frac{fm_1}{a_1^3}}. \end{aligned} \quad (1.10)$$

$$\begin{aligned} \mathbf{R}_2 &= R_2 \boldsymbol{\eta}_2, \quad \boldsymbol{\eta}_2 = (\eta_{2x}, \eta_{2y}, \eta_{2z}), \\ \eta_{2x} &= \cos h_2 \cos(g_2 + \vartheta_2) - \sin h_2 \cos i_2 \sin(g_2 + \vartheta_2), \\ \eta_{2y} &= \sin h_2 \cos(g_2 + \vartheta_2) + \cos h_2 \cos i_2 \sin(g_2 + \vartheta_2), \\ \eta_{2z} &= \sin i_2 \sin(g_2 + \vartheta_2). \end{aligned} \quad (1.11)$$

In the limited setting under consideration, the first term on the right-hand side of formula (1.6) is a constant. Therefore, we can write the Hamiltonian of the problem in the form:

$$\mathcal{H} = -\frac{f^2(m_2 + m_3)^2 m_r^3}{2L_2^2} + \mathcal{H}_1. \quad (1.12)$$

Taking into account (1.7)–(1.11), the perturbing part of the Hamiltonian \mathcal{H}_1 in formula (1.12) will take the following form:

$$\mathcal{H}_1 = \frac{f^2 m_1^4 G_2^4 (m_2 + m_3)^4 (1 + e_1 \cos \vartheta_1)^3}{2G_1^6 m_r^3 (1 + e_2 \cos \vartheta_2)^2} (1 - 3 \cos^2 \psi_{12}), \quad (1.13)$$

$$\begin{aligned} \cos \psi_{12} = (\mathbf{n}_1, \mathbf{n}_2) &= \cos(g_1 + \vartheta_1) [\cos h_2 \cos(g_2 + \vartheta_2) - \sin h_2 \cos i_2 \sin(g_2 + \vartheta_2)] + \\ &+ \sin(g_1 + \vartheta_1) [\sin h_2 \cos(g_2 + \vartheta_2) + \cos h_2 \cos i_2 \sin(g_2 + \vartheta_2)]. \end{aligned}$$

2. EVOLUTION OF SATELLITE ORBITAL MOTION

Let us perform the procedure of averaging the Hamiltonian over the “fast” angular variables—the mean anomalies l_1, l_2 :

$$\langle * \rangle_{l_1, l_2} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} (*) dl_1 dl_2 = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} (*) \frac{(1 - e_1^2)^{3/2}}{(1 + e_1 \cos \vartheta_1)^2} \cdot \frac{(1 - e_2^2)^{3/2}}{(1 + e_2 \cos \vartheta_2)^2} d\vartheta_1 d\vartheta_2.$$

The perturbing part of the Hamiltonian (1.13) will take the following form as a result of the averaging procedure:

$$\langle \mathcal{H}_1 \rangle_{l_1, l_2} = \frac{f^2 m_1^4 (m_2 + m_3)^4}{16m_r^3} \cdot \frac{L_2^4}{G_1^3 L_1^3} \left\{ 2(1 - 3 \cos^2 i_2) + 3e_2^2 (1 - 3 \cos^2 i_2 - 5 \cos(2g_2) \sin^2 i_2) \right\}. \quad (2.1)$$

Expressing eccentricity e_2 and inclination i_2 through the variables L_2, G_2, H_2 according to formulas (1.8), (1.9) and taking into account formulas (1.12), (2.1), we will move on to the averaged Hamiltonian in Delaunay variables:

$$\langle \mathcal{H} \rangle_{l_1, l_2} = -\frac{f^2(m_2 + m_3)^2 m_r^3}{2L_2^2} + \frac{f^2 m_1^4 (m_2 + m_3)^4}{16m_r^3} \cdot \frac{L_2^4}{G_1^3 L_1^3} \left\{ \left(1 - 3 \frac{H_2^2}{G_2^2} \right) \left(5 - 3 \frac{G_2^2}{L_2^2} \right) - 15 \cos(2g_2) \left(1 - \frac{H_2^2}{G_2^2} \right) \left(1 - \frac{G_2^2}{L_2^2} \right) \right\}. \quad (2.2)$$

Thus, we obtain an average representation of the Hamiltonian in terms of fast angular variables. The averaged equations of motion are written as a system of canonical equations:

$$\begin{aligned} \dot{L}_2 &= -\frac{\partial \langle \mathcal{H}_1 \rangle_{l_1, l_2}}{\partial l_2}, \quad \dot{G}_2 = -\frac{\partial \langle \mathcal{H}_1 \rangle_{l_1, l_2}}{\partial g_2}, \quad \dot{H}_2 = -\frac{\partial \langle \mathcal{H}_1 \rangle_{l_1, l_2}}{\partial h_2}, \quad \dot{l}_2 = \frac{\partial \langle \mathcal{H}_1 \rangle_{l_1, l_2}}{\partial L_2} + \frac{f^2(m_2 + m_3)^2 m_r^3}{L_2^3}, \\ \dot{g}_2 &= \frac{\partial \langle \mathcal{H}_1 \rangle_{l_1, l_2}}{\partial G_2}, \quad \dot{h}_2 = \frac{\partial \langle \mathcal{H}_1 \rangle_{l_1, l_2}}{\partial H_2}. \end{aligned} \quad (2.3)$$

Taking into account formula (2.2), the evolutionary system of equations of motion of the satellite in Delaunay variables (2.3) has the form:

$$\begin{aligned} \dot{L}_2 &= 0, \quad \dot{H}_2 = 0, \\ \dot{G}_2 &= -\frac{15 f^2 m_1^4 (m_2 + m_3)^4}{8m_r^3} \cdot \frac{L_2^4}{G_1^3 L_1^3} \left(1 - \frac{H_2^2}{G_2^2} \right) \left(1 - \frac{G_2^2}{L_2^2} \right) \sin(2g_2), \end{aligned}$$

$$\begin{aligned} \dot{g}_2 &= \frac{6f^2 m_1^4 (m_2 + m_3)^4}{16m_1^3} \cdot \frac{L_2^4}{G_1^3 L_1^3 G_2} \left\{ 5 \frac{H_2^2}{G_2^2} (1 - \cos(2g_2)) - \frac{G_2^2}{L_2^2} (1 - 5 \cos(2g_2)) \right\}, \\ \dot{h}_2 &= \frac{-6f^2 m_1^4 (m_2 + m_3)^4}{16m_1^3 G_2} \cdot \frac{L_2^4}{G_1^3 L_1^3} \left(\frac{H_2}{G_2} \right) \left\{ 2 + \left(1 - \frac{G_2^2}{L_2^2} \right) (3 - 5 \cos(2g_2)) \right\}, \\ \dot{i}_2 &= \frac{f^2 (m_2 + m_3)^2 m_r^3}{L_2^3} + \frac{f^2 m_1^4 (m_2 + m_3)^4}{16m_1^3 G_1^3 L_1^3} 2L_2^3 \left\{ \left(1 - 3 \frac{H_2^2}{G_2^2} \right) \left(10 - 3 \frac{G_2^2}{L_2^2} \right) - 15 \cos(2g_2) \left(1 - \frac{H_2^2}{G_2^2} \right) \left(2 - \frac{G_2^2}{L_2^2} \right) \right\}. \end{aligned}$$

Using the following inverse ratios $H_2^2/G_2^2 = \cos^2 i_2$, $1 - (H_2^2/G_2^2) = \sin^2 i_2$, $G_2^2/L_2^2 = 1 - e_2^2$, obtained from equations (1.8) and (1.9), we derive an averaged system of equations for the satellite's motion in Delaunay variables (relative to "slow" variables):

$$\begin{aligned} \dot{L}_2 &= 0, \quad \dot{H}_2 = 0, \\ \dot{G}_2 &= -\frac{15 f m_1 m_r}{8} \cdot \frac{a_2^2 e_2^2}{a_1^3 (1 - e_1^2)^{3/2}} \sin^2 i_2 \sin(2g_2), \\ \dot{i}_2 &= \frac{\sqrt{f(m_2 + m_3)}}{a_2^{3/2}} + \frac{m_1 \sqrt{f}}{8 \sqrt{(m_2 + m_3)}} \cdot \frac{a_2^{3/2}}{a_1^3 (1 - e_1^2)^{3/2}} \left\{ (-2 + 3 \sin^2 i_2)(7 + 3e_2^2) - 15 \cos(2g_2) \sin^2 i_2 (1 + e_2^2) \right\}, \quad (2.4) \\ \dot{g}_2 &= \frac{3m_1 \sqrt{f}}{8 \sqrt{(m_2 + m_3)}} \cdot \frac{a_2^{3/2}}{a_1^3 (1 - e_1^2)^{3/2} \sqrt{(1 - e_2^2)}} \left\{ 4 + e_2^2 (1 - 5 \cos(2g_2)) - 5 \sin^2 i_2 (1 - \cos(2g_2)) \right\}, \\ \dot{h}_2 &= \frac{-3m_1 \sqrt{f}}{8 \sqrt{(m_2 + m_3)}} \cdot \frac{a_2^{3/2}}{a_1^3 (1 - e_1^2)^{3/2} \sqrt{(1 - e_2^2)}} \cos i_2 \left\{ 2 + e_2^2 (3 - 5 \cos(2g_2)) \right\}. \end{aligned}$$

Next, we will perform transformations in the evolutionary system of equations of motion to transition to a form that depends on the variables e_2 , i_2 , g_2 , h_2 , and also perform a transition to dimensionless time $\tau = n_2 t$, $n_2 = \sqrt{f(m_2 + m_3)} a_2^{-3/2}$. The final system takes the form:

$$\begin{aligned} a_2' &= 0, \\ e_2' &= \frac{15 \mu a_0^3}{8} \cdot \frac{e_2 \sqrt{1 - e_2^2}}{(1 - e_1^2)^{3/2}} \sin^2 i_2 \sin(2g_2), \\ i_2' &= \frac{-15 \mu}{16} \cdot \frac{a_0^3 e_2^2}{\sqrt{1 - e_2^2} (1 - e_1^2)^{3/2}} \sin(2i_2) \sin(2g_2), \\ l_2' &= 1 + \frac{\mu}{8} \cdot \frac{a_0^3}{(1 - e_1^2)^{3/2}} \left\{ (-2 + 3 \sin^2 i_2)(7 + 3e_2^2) - 15 \cos(2g_2) \sin^2 i_2 (1 + e_2^2) \right\}, \quad (2.5) \\ g_2' &= \frac{3 \mu}{8} \cdot \frac{a_0^3}{(1 - e_1^2)^{3/2} \sqrt{1 - e_2^2}} \left\{ 4 + e_2^2 (1 - 5 \cos(2g_2)) - 5 \sin^2 i_2 (1 - \cos(2g_2)) \right\}, \\ h_2' &= \frac{-3 \mu}{8} \cdot \frac{a_0^3}{(1 - e_1^2)^{3/2} \sqrt{1 - e_2^2}} \cos i_2 \left\{ 2 + e_2^2 (3 - 5 \cos(2g_2)) \right\}, \end{aligned}$$

where $\mu = \frac{m_1}{m_2 + m_3}$, $a_0^3 = \frac{a_2^3}{a_1^3}$ is the mass ratio (the ratio of the mass of the attracting center to the sum of the masses of the celestial body and the satellite) and the axial ratio (the ratio of the semi-major axes of the orbits), respectively.

Note that the value of the major semi-axis a_2 does not change over time. Furthermore, the right-hand sides of the system of equations (2.5) depend only on three variables: eccentricity e_2 , orbital inclination i_2 , and longitude of the pericenter g_2 . Therefore, the 2nd, 3rd, and 5th equations of the system are separated from the rest of the equations, and the 4th and 6th equations can be integrated after the functions $e_2(\tau)$, $i_2(\tau)$, $g_2(\tau)$ are found. The angular variable l_2 is “fast.” This variable was averaged. Therefore, it can be excluded from consideration. The system is reduced to four basic equations:

$$\begin{aligned} e_2' &= \frac{15k}{8} \cdot \frac{e_2 \sqrt{1-e_2^2}}{(1-e_1^2)^{\frac{3}{2}}} \sin^2 i_2 \sin(2g_2), \\ i_2' &= \frac{-15k}{16} \cdot \frac{e_2^2}{\sqrt{1-e_2^2} (1-e_1^2)^{\frac{3}{2}}} \sin(2i_2) \sin(2g_2), \\ g_2' &= \frac{3k}{8} \cdot \frac{1}{(1-e_1^2)^{\frac{3}{2}} \sqrt{1-e_2^2}} \left\{ 4 + e_2^2 (1 - 5 \cos(2g_2)) - 5 \sin^2 i_2 (1 - \cos(2g_2)) \right\}, \\ h_2' &= \frac{-3k}{8} \cdot \frac{\cos i_2}{(1-e_1^2)^{\frac{3}{2}} \sqrt{1-e_2^2}} \left\{ 2 + e_2^2 (3 - 5 \cos(2g_2)) \right\}, \end{aligned} \quad (2.6)$$

where $k = \mu a_0^3$.

The resulting system (2.6), with an accuracy up to a constant factor responsible for the choice of dimensionless time, coincides with the system of equations obtained by M.L. Lidov [7, 8], if we neglect the perturbations caused by the eccentricity of the Moon’s gravitational field. The system of equations (2.6) has first integrals [7, 8]:

$$\cos^2 i_2 (1 - e_2^2) = C, \quad (2.7)$$

$$e_2^2 \left(\frac{2}{5} - \sin^2 g_2 \sin^2 i_2 \right) = D. \quad (2.8)$$

From the resulting ratio (2.7), it follows that as the value of e_2 increases, the value of i_2 will decrease, and *vice versa*. Let us move on to the parameters of the system (2.6):

$m_1 = 5.9736 \cdot 10^{24}$ kg is the mass of the Earth,

$m_2 = 7.349 \cdot 10^{22}$ kg is the mass of the Moon,

$m_3 = 10^3$ kg is the mass of the satellite,

$$\mu = \frac{m_1}{m_2 + m_3} = \frac{5.9736 \cdot 10^{24}}{7.349 \cdot 10^{22} + 10^3} \approx 81.28453.$$

As a first approximation, we can assume that the Moon moves in an elliptical orbit with eccentricity $e_1 = 0.0549$ and a semi-major axis of the geocentric orbit $a_1 = 384400 \cdot 10^3$ m.

We will consider lunar orbits with altitudes ranging from 500 to 20000 km (310 to 12430 miles), where the Earth’s gravity has the greatest effect and causes orbital perturbations. Let $a_2 = 6500 \cdot 10^3$ m be the semi-major axis of the satellite’s orbit, then:

$$a_0^3 = \frac{a_2^3}{a_1^3} = \frac{6500 \cdot 10^3}{379739 \cdot 10^3} \approx 0.017117, \quad k = \mu a_0^3 \approx 81.28453 \cdot 0.017117 \approx 1.39135.$$

We can integrate the resulting system (2.6) using the fourth-order Runge–Kutta method with a constant step size, setting the initial values: $e_{2_0} = 0.01$, $i_{2_0} = \frac{j\pi}{180}$, $g_{2_0} = 0$, $h_{2_0} = 0$, $j = 30^\circ, 45^\circ, 60^\circ, 85^\circ$, and on this basis we can construct integral curves. The resulting graphs (Fig. 3) clearly demonstrate compliance with the relationship of the first integral (2.7) of the system under consideration. The change in the value of eccentricity (the degree of ellipticity of the orbit) and the inclination angle (the angle between the orbital plane and a certain reference plane) on the graph can be either gradual or abrupt.

3. STATIONARY SOLUTIONS OF THE EVOLUTIONARY SYSTEM OF MOTION EQUATIONS OF THE ASM

Frozen orbits around the Moon, natural satellites, or asteroids are of great interest, since a number of space missions aim to fly around such bodies. Frozen orbits are those in which changes in inclination, eccentricity, and longitude of the pericenter from the ascending node are minimized. In order to define such orbits, we will determine stationary solutions to the first three equations (2.6), determined by the conditions:

$$e'_2 = 0, i'_2 = 0, g'_2 = 0. \quad (3.1)$$

Since elliptical orbits are being considered, we will limit the values of eccentricity e_2 to the interval $[0; 1)$:

$$0 \leq e_2 < 1. \quad (3.2)$$

Considering integral (2.7), we obtain a closed autonomous system of the second order with respect to g_2, e_2 :

$$\begin{aligned} e'_2 &= 5k_1 e_2 \sqrt{1 - e_2^2} (1 - e_2^2 - C) \sin(2g_2), \\ g'_2 &= \frac{k_1}{(1 - e_2^2)^{3/2}} \left\{ (1 - e_2^2) (4 + e_2^2 - 5e_2^2 \cos(2g_2)) - 5(1 - e_2^2 - C)(1 - \cos(2g_2)) \right\}. \end{aligned} \quad (3.3)$$

Here

$$k_1 = \frac{3k}{8(1 - e_1^2)^{3/2}}.$$

Let us find stationary solutions to system (3.3) by setting its right-hand sides equal to zero. Taking into account condition (3.2), we obtain the following system of equations:

$$\begin{aligned} e_2 (1 - e_2^2 - C) \sin(2g_2) &= 0, \\ (1 - e_2^2) (4 + e_2^2 - 5e_2^2 \cos(2g_2)) - 5(1 - e_2^2 - C)(1 - \cos(2g_2)) &= 0. \end{aligned} \quad (3.4)$$

Note that according to equality (2.7), the value of the constant C is limited by the interval $[0; 1]$.

From the first equation of system (3.4), it follows that either $e_2 = 0$, or $1 - e_2^2 - C = 0$, or $\sin(2g_2) = 0$. In the case $e_2 = 0$ from the second equation of system (3.4), we obtain:

$$\cos(2g_2) = \frac{1 - 5C}{5(1 - C)}. \quad (3.5)$$

Equation (3.5) has a solution under the condition $0 \leq C \leq 0.6$. Therefore, with the specified values of the parameter C we obtain the first series of stationary solutions:

$$e_2^* = 0, \quad g_2^* = \pm \frac{1}{2} \arccos \frac{1 - 5C}{5(1 - C)} + \pi m, \quad m \in \mathbb{Z} \quad (C \in [0; 0.6]). \quad (3.6)$$

If $1 - e_2^2 - C = 0$, then from the second equation of system (3.4) we obtain:

$$\cos(2g_2) = 1 + \frac{4C}{5(1 - C)}.$$

Given the restrictions imposed on the constant C , the latter equality is possible if $C = 0$. But then $e_2 = 1$. Thus we come at a contradiction with (3.2).

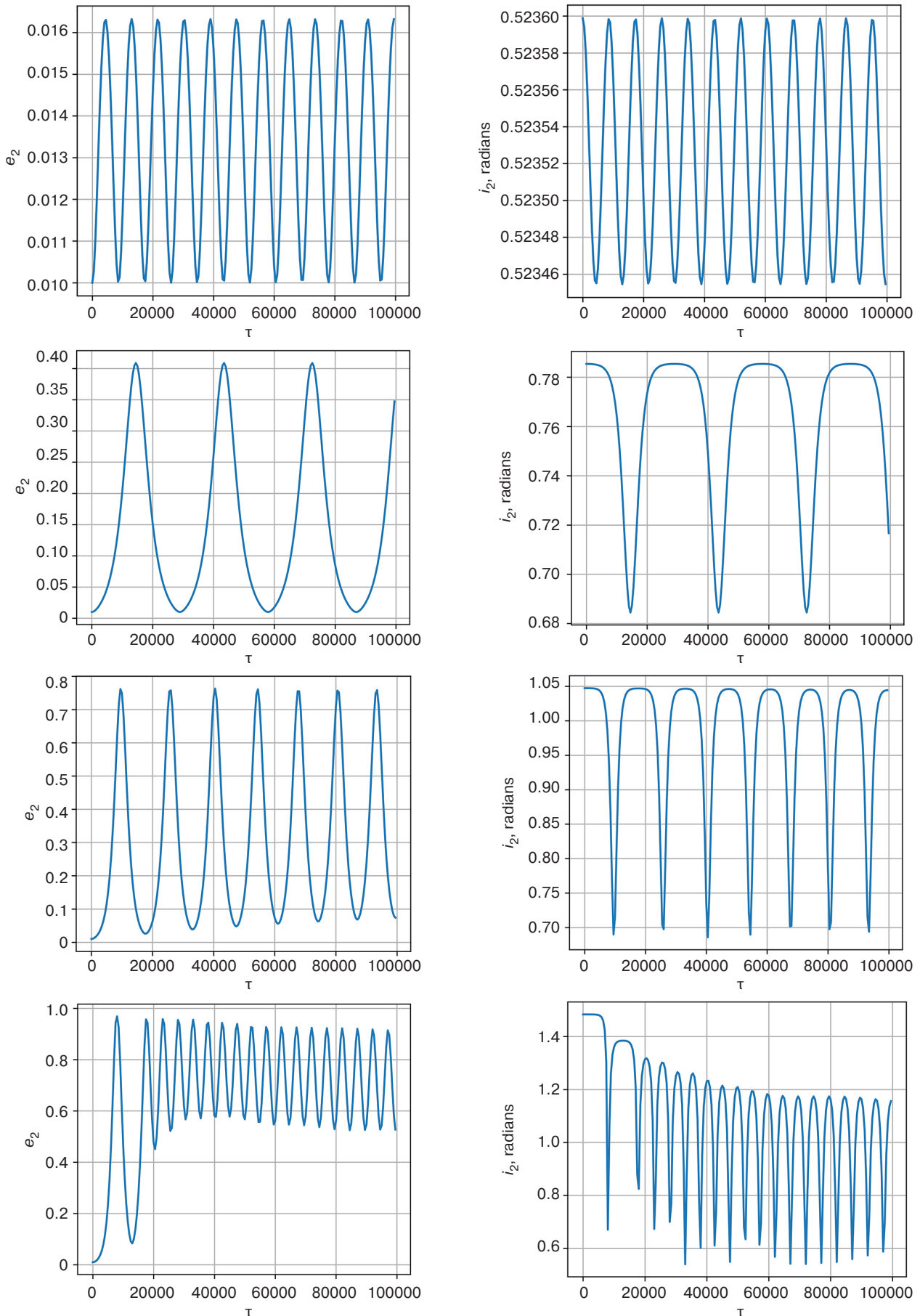


Fig. 3. Integral curves of the averaged system of equations (2.6) $e_2(\tau)$ and $i_2(\tau)$ at $i_{2_0} = \pi/6, \pi/4, \pi/3, 85\pi/180$

If $\sin(2g_2) = 0$, then either $\cos(2g_2) = 1$, or $\cos(2g_2) = -1$. If $\cos(2g_2) = 1$, then from the second equation of system (3.4) we obtain $e_2 = 1$, therefore this case is not suitable. If $\cos(2g_2) = -1$, then from the second equation of system (3.4) it follows that:

$$e_2^2 = 1 - \sqrt{\frac{5C}{3}}.$$

This equality is possible under the condition: $0 < C \leq 0.6$. Thus, we obtained another series of stationary solutions:

$$e_2^* = \sqrt{1 - \sqrt{\frac{5C}{3}}}, \quad g_2^* = \frac{\pi}{2} + \pi m, \quad m \in \mathbb{Z} \quad (C \in (0; 0.6]). \quad (3.7)$$

Let us investigate the stability of stationary solutions based on a perturbed system of first-order equations. We can denote $e_2 = e_2^* + x_1$, $g_2 = g_2^* + x_2$, and further denote the right-hand sides of the system of equations (3.3) respectively as $F_1(e_2, g_2)$, $F_2(e_2, g_2)$. Then the perturbed system of equations of the first approximation will be written as:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2, & x_2' &= a_{21}x_1 + a_{22}x_2, \\ a_{11} &= \frac{\partial F_1(e_2^*, g_2^*)}{\partial e_2}, & a_{12} &= \frac{\partial F_1(e_2^*, g_2^*)}{\partial g_2}, & a_{21} &= \frac{\partial F_2(e_2^*, g_2^*)}{\partial e_2}, & a_{22} &= \frac{\partial F_2(e_2^*, g_2^*)}{\partial g_2}. \end{aligned} \quad (3.8)$$

The partial derivatives of the right-hand sides of the evolutionary system of equations (3.3) are of the form:

$$\begin{aligned} \frac{\partial F_1(e_2, g_2)}{\partial e_2} &= \frac{5k_1 \sin(2g_2)}{\sqrt{1-e_2^2}} \left\{ (1-e_2^2)(1-3e_2^2-C) - e_2^2(1-e_2^2-C) \right\}, \\ \frac{\partial F_1(e_2, g_2)}{\partial g_2} &= 10k_1 e_2 \sqrt{1-e_2^2} (1-e_2^2-C) \cos(2g_2), \\ \frac{\partial F_2(e_2, g_2)}{\partial e_2} &= \frac{3e_2 F_2(e_2, g_2)}{1-e_2^2} + \frac{4k_1 e_2}{\sqrt{1-e_2^2}} (1-5 \cos(2g_2)), \\ \frac{\partial F_2(e_2, g_2)}{\partial g_2} &= \frac{10k_1 \sin(2g_2)}{(1-e_2^2)^{3/2}} \left(C - (1-e_2^2)^2 \right). \end{aligned}$$

For the stationary solution (3.6), the coefficients a_{ij} on the right-hand sides of equations (3.8) are as follows:

$$a_{11} = 5k_1 \sin(2g_2^*)(1-C), \quad a_{12} = 0, \quad a_{21} = 0, \quad a_{22} = -2a_{11}.$$

The roots of the characteristic equation are real and of different signs (the type of singular point is a saddle point). Consequently, the stationary solution (3.6) is unstable.

For the stationary solution (3.7), we obtain:

$$a_{11} = 0, \quad a_{12} = -10k_1 e_2^* \sqrt{1-e_2^{*2}} \left(\sqrt{\frac{5C}{3}} - C \right) < 0, \quad a_{21} = \frac{24k_1 e_2^{*2}}{\sqrt{1-e_2^{*2}}} > 0, \quad a_{22} = 0.$$

The roots of the characteristic equation $\lambda^2 - a_{12}a_{21} = 0$ are purely imaginary. Therefore, in the first approximation, the stationary solution is stable (the type of singular point is a center).

Figures 4–7 show phase portraits of the system of equations (3.3) for different values of the constant C .

We can see that the nature of the change in eccentricity is oscillatory. With regard to the evolution of the argument of perihelion, its change demonstrates either a system of open trajectories of a monotonic nature or a system of closed trajectories of an oscillatory nature surrounding the equilibrium positions. According to the analytical study, when $C > 0.6$ the system of equations (3.3) has no stationary solutions. This case is shown in Fig. 7. Here, the changes in eccentricity are oscillatory, and the longitude of the pericenter changes monotonically.

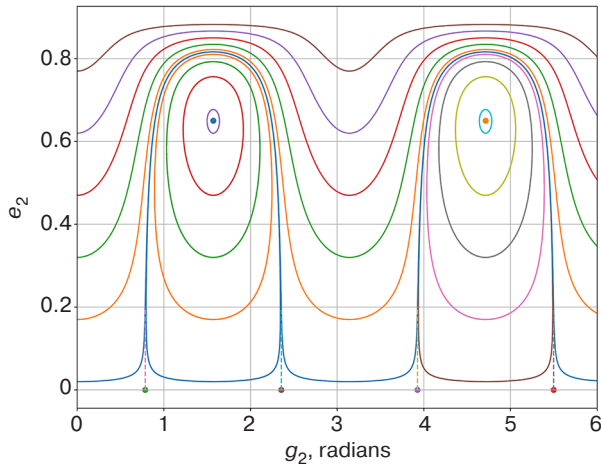


Fig. 4. Phase portrait of the system of equations (3.3) at $C = 0.2$

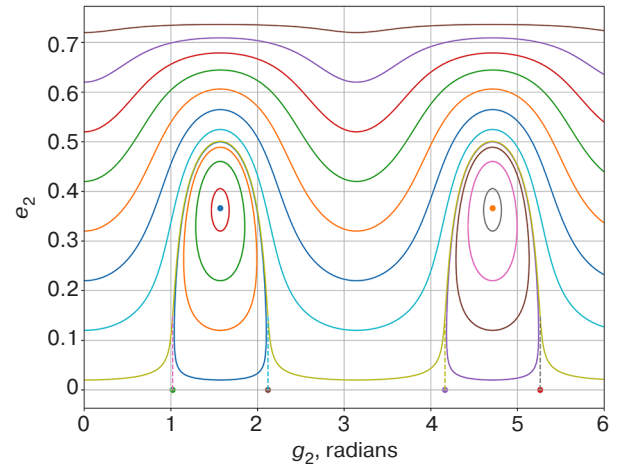


Fig. 5. Phase portrait of the system of equations (3.3) at $C = 0.45$

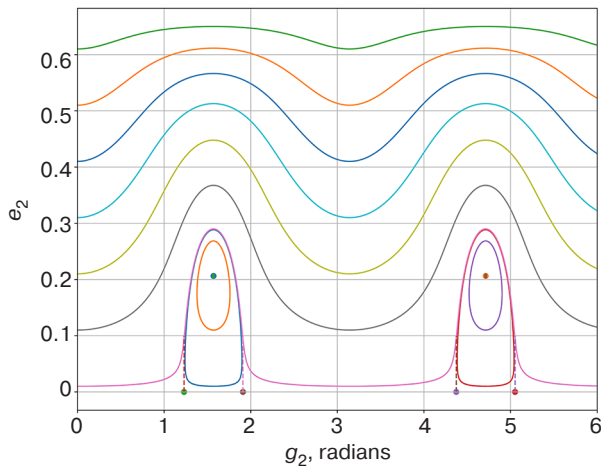


Fig. 6. Phase portrait of the system of equations (3.3) at $C = 0.55$

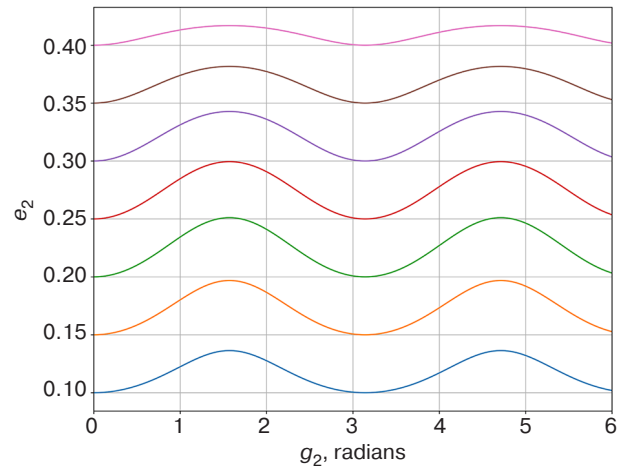


Fig. 7. Phase portrait of the system of equations (3.3) at $C = 0.8$

4. UNMEDIATED EQUATIONS OF SATELLITE MOTION FOR THE EXCITED TASK

Let us derive the unaveraged system of equations of motion for the ASM. The Hamiltonian of the problem has the form:

$$\mathcal{H} = -\frac{f^2(m_2 + m_3)^2 m_r^3}{2L_2^2} + \mathcal{H}_1, \quad \mathcal{H}_1 = -\frac{1}{2} \cdot \frac{f m_1 m_r R_2^2}{R_1^3} (3\Phi^2 - 1). \quad (4.1)$$

The summand \mathcal{H}_1 in the expression for the Hamiltonian depends on the Delaunay variables $L_2, G_2, H_2, l_2, g_2, h_2$ via $R_2 = R_2(L_2, G_2, l_2)$ according to (1.8) and Φ :

$$\Phi = \cos \psi_{12} = \cos(g_2 + \vartheta_2) \cos(g_1 + \vartheta_1 - h_2) + \cos i_2 \sin(g_2 + \vartheta_2) \sin(g_1 + \vartheta_1 - h_2). \quad (4.2)$$

The canonical equations of motion of a satellite in Delaunay variables are as follows:

$$\dot{L}_2 = -\frac{\partial \mathcal{H}_1}{\partial l_2}, \quad \dot{G}_2 = -\frac{\partial \mathcal{H}_1}{\partial g_2}, \quad \dot{H}_2 = -\frac{\partial \mathcal{H}_1}{\partial h_2}, \quad \dot{l}_2 = \frac{f^2(m_2 + m_3)^2 m_r^3}{L_2^3} + \frac{\partial \mathcal{H}_1}{\partial L_2}, \quad \dot{g}_2 = \frac{\partial \mathcal{H}_1}{\partial G_2}, \quad \dot{h}_2 = \frac{\partial \mathcal{H}_1}{\partial H_2}. \quad (4.3)$$

After calculating the partial derivatives, we obtain:

$$\begin{aligned} \dot{L}_2 &= \frac{fm_1 m_r R_2^2}{R_1^3} \cdot \frac{(1+e_2 \cos \vartheta_2)}{(1-e_2^2)^{3/2}} \left\{ (3\Phi^2 - 1)e_2 \sin \vartheta_2 + 3(1+e_2 \cos \vartheta_2)\Phi\Phi_2 \right\}, \\ \dot{G}_2 &= \frac{3fm_1 m_r R_2^2}{R_1^3} \Phi\Phi_2, \quad \dot{H}_2 = \frac{3fm_1 m_r R_2^2}{R_1^3} \Phi\Phi_3, \\ \dot{l}_2 &= n_2 - \frac{fm_1 m_r R_2^2}{R_1^3 L_2 e_2} \left\{ (2e_2 - \cos \vartheta_2 - e_2 \cos^2 \vartheta_2)(3\Phi^2 - 1) + 3\Phi\Phi_2 \sin \vartheta_2 (2 + e_2 \cos \vartheta_2) \right\}, \\ \dot{g}_2 &= \frac{fm_1 m_r R_2^2}{R_1^3 e_2 G_2} \left\{ (1+e_2 \cos \vartheta_2) \cos \vartheta_2 (1-3\Phi^2) + 3\Phi\Phi_2 \sin \vartheta_2 (2 + e_2 \cos \vartheta_2) + \right. \\ &\quad \left. + 3\Phi e_2 \cos i_2 \sin(g_2 + \vartheta_2) \sin(g_1 + \vartheta_1 - h_2) \right\}, \\ \dot{h}_2 &= -\frac{3fm_1 m_r R_2^2}{R_1^3 G_2} \Phi \sin(g_2 + \vartheta_2) \sin(g_1 + \vartheta_1 - h_2). \end{aligned} \quad (4.4)$$

Here

$$\begin{aligned} n_2 &= \frac{f^2 (m_2 + m_3)^2 m_r^3}{L_2^3}, \\ \Phi_2 &= \frac{\partial \Phi}{\partial \vartheta_2} = -\sin(g_2 + \vartheta_2) \cos(g_1 + \vartheta_1 - h_2) + \cos i_2 \cos(g_2 + \vartheta_2) \sin(g_1 + \vartheta_1 - h_2), \\ \Phi_3 &= \frac{\partial \Phi}{\partial h_2} = \cos(g_2 + \vartheta_2) \sin(g_1 + \vartheta_1 - h_2) - \cos i_2 \sin(g_2 + \vartheta_2) \cos(g_1 + \vartheta_1 - h_2). \end{aligned} \quad (4.5)$$

Thus, we obtain a system of equations (3.4) for the motion of a 6th-order satellite in Delaunay variables. The first equation with \dot{L}_2 is responsible for the change in total orbital action over time, the second with \dot{G}_2 is responsible for the change in total angular momentum over time, and the third with \dot{H}_2 is responsible for the change in azimuthal angular momentum over time. The remaining three equations are responsible for changes in the orbital elements: mean anomaly l_2 , argument of pericenter g_2 ; and the longitude of ascending node h_2 . Point C_0 moves along an undisturbed Keplerian elliptical orbit in the OXY plane. Therefore R_1, ϑ_1 in (4.4), (4.5) are given functions of time according to (1.10), and the longitude of the pericenter g_1 , the semi-major axis a_1 and the eccentricity e_1 are constant values.

Let us make the transition to dimensionless variables $n_{20}, e_2, i_2, g_2, \vartheta_2, h_2$, where

$$n_{20} = \frac{n_2}{n_1} = \frac{f^2 (m_2 + m_3)^2 m_r^3}{L_2^3 n_1}, \quad n_1 = \sqrt{\frac{fm_1}{a_1^3}}.$$

Furthermore, let us move on to dimensionless time $\tau = n_1 t / (2\pi)$ the number of revolutions of the center of mass C_0 relative to point O . We will transform the right-hand sides of the equations of system (4.4), taking into account (1.10) and relying on the known relations for the Delaunay variables:

$$\begin{aligned} L_2 &= m_r \sqrt{f(m_2 + m_3)a_2}, \quad G_2 = m_r \sqrt{f(m_2 + m_3)a_2(1-e_2^2)}, \\ H_2 &= m_r \sqrt{f(m_2 + m_3)a_2(1-e_2^2)} \cos i_2. \end{aligned} \quad (4.6)$$

As a result, we obtain:

$$n'_{20} = -\frac{6\pi \sqrt{1-e_2^2} (1+e_1 \cos \vartheta_1)^3}{(1-e_1^2)^3 (1+e_2 \cos \vartheta_2)} \left\{ (3\Phi^2 - 1)e_2 \sin \vartheta_2 + 3(1+e_2 \cos \vartheta_2)\Phi\Phi_2 \right\},$$

$$\begin{aligned}
 e_2' &= \frac{2\pi\sqrt{(1-e_2^2)^3}(1+e_1\cos\vartheta_1)^3}{(1-e_1^2)^3(1+e_2\cos\vartheta_2)^2 n_{20}} \left\{ (3\Phi^2 - 1)\sin\vartheta_2(1+e_2\cos\vartheta_2) + 3\Phi\Phi_2(2\cos\vartheta_2 + e_2\cos^2\vartheta_2 + e_2) \right\}, \\
 i_2' &= -\frac{6\pi\sqrt{(1-e_2^2)^3}(1+e_1\cos\vartheta_1)^3}{(1-e_1^2)^3(1+e_2\cos\vartheta_2)^2 n_{20}} \Phi \sin i_2 \cos(g_2 + \vartheta_2) \sin(g_1 + \vartheta_1 - h_2), \\
 g_2' &= \frac{2\pi\sqrt{(1-e_2^2)^3}(1+e_1\cos\vartheta_1)^3}{(1-e_1^2)^3(1+e_2\cos\vartheta_2)^2 e_2 n_{20}} \left\{ (1+e_2\cos\vartheta_2)\cos\vartheta_2(1-3\Phi^2) + \right. \\
 &\quad \left. + 3\Phi\Phi_2 \sin\vartheta_2(2+e_2\cos\vartheta_2) + 3\Phi e_2 \cos i_2 \sin(g_2 + \vartheta_2) \sin(g_1 + \vartheta_1 - h_2) \right\}, \\
 \vartheta_2' &= \frac{2\pi(1+e_2\cos\vartheta_2)^2 n_{20}}{(1-e_2^2)^{3/2}} + \\
 &\quad + \frac{2\pi\sqrt{(1-e_2^2)^3}(1+e_1\cos\vartheta_1)^3}{(1-e_1^2)^3(1+e_2\cos\vartheta_2)^2 e_2 n_{20}} \left\{ (1+e_2\cos\vartheta_2)\cos\vartheta_2(3\Phi^2 - 1) - 3\Phi\Phi_2 \sin\vartheta_2(2+e_2\cos\vartheta_2) \right\}, \\
 h_2' &= -\frac{6\pi\sqrt{(1-e_2^2)^3}(1+e_1\cos\vartheta_1)^3}{(1-e_1^2)^3(1+e_2\cos\vartheta_2)^2 n_{20}} \Phi \sin(g_2 + \vartheta_2) \sin(g_1 + \vartheta_1 - h_2), \\
 \vartheta_1' &= \frac{2\pi(1+e_1\cos\vartheta_1)^2}{(1-e_1^2)^{3/2}}.
 \end{aligned} \tag{4.7}$$

We will perform the numerical integration of the averaged and unaveraged systems of equations (2.6) and (4.7) and compare the obtained numerical results with the modern temporal variations of eccentricity and inclination presented in [12]. As initial values, following [12], for the averaged system of equations (2.6) we take (2.6)

$$e_2(0) = 0.61, i_2(0) = 55.9^\circ = 55.9\pi/180 \text{ rad}, g_2(0) = \pi/2, h_2(0) = 0. \tag{4.8}$$

Since a dimensionless time in system (2.6), $\tau = n_2 t$, $n_2 = \sqrt{f(m_2 + m_3)} a_2^{-3}$ is used, therefore, in order to consider a double period $2T = 2.5$ years or $2.5 \cdot 365$ days, the time range needs to be set: $\tau = \frac{2.5 \cdot 365}{0.0874} \cong 10440$. Figure 8 shows the integral curves of the averaged system of equations (2.6).

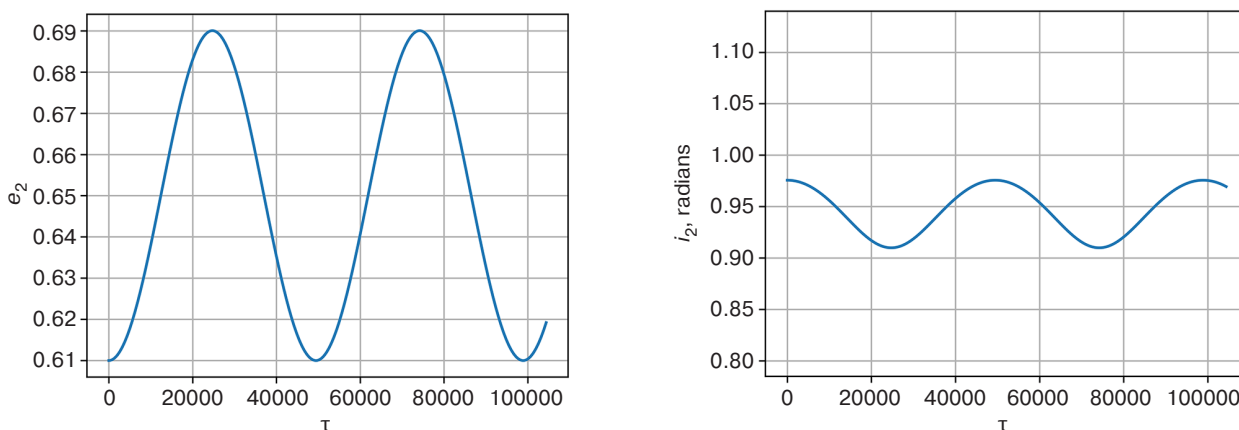


Fig. 8. Integral curves $e_2(\tau)$ and $i_2(\tau)$ of the averaged system of equations

We obtain the following predicted deviations of the average values of the elements: $\Delta e_2 \sim 0.08$, $\Delta i_2 \sim 3.7^\circ$ (0.07 rad), close to the results from [12]. This reflects the basic nature of the changes in the elements, while omitting short-period gravitational effects.

Next, we integrate the unaveraged system (4.7). In order to do this, we need to specify the initial values of the eccentricity, semi-major axis, longitude of the pericenter of the Moon’s orbit, and the initial value of its true anomaly. The data was collected using the *Horizons System*⁴ utility with the following parameters: target object is the Moon; coordinate center is the Earth; and time is 2009-07-15 01:00:00. Since [12] provides only 4 initial values (4.8), the remaining missing values of the true and mean anomaly were also collected using the *Horizons System* for the ASM operating at that time: LRO 2009-031A⁵. As a dimensionless time in the system (4.7), $\tau = n_1 t / (2\pi)$ is used. Therefore, in order to consider the double period $2T = 2.5$ years or $2.5 \cdot 365$ days, the time range needs to be set in $\tau^* = \frac{2.5 \cdot 365}{27.0632} \cong 34$.

Figure 9 shows the integral curves $e_2(t)$ and $i_2(t)$ of the averaged and non-averaged systems of equations. Figure 10 shows the integral curves of the systems of equations (2.6) and (4.7) with the same initial conditions in the plane (g_2, e_2) , which are generally similar to the analogous curves presented in [12]. A characteristic steady oscillatory motion is observed, the length of the pericenter depends on the eccentricity in accordance with the periodic law. A change in the eccentricity e_2 is noted in the range from 0.61 to 0.69, which is relatively similar to $\Delta e \sim 0.09$ in the work [12].

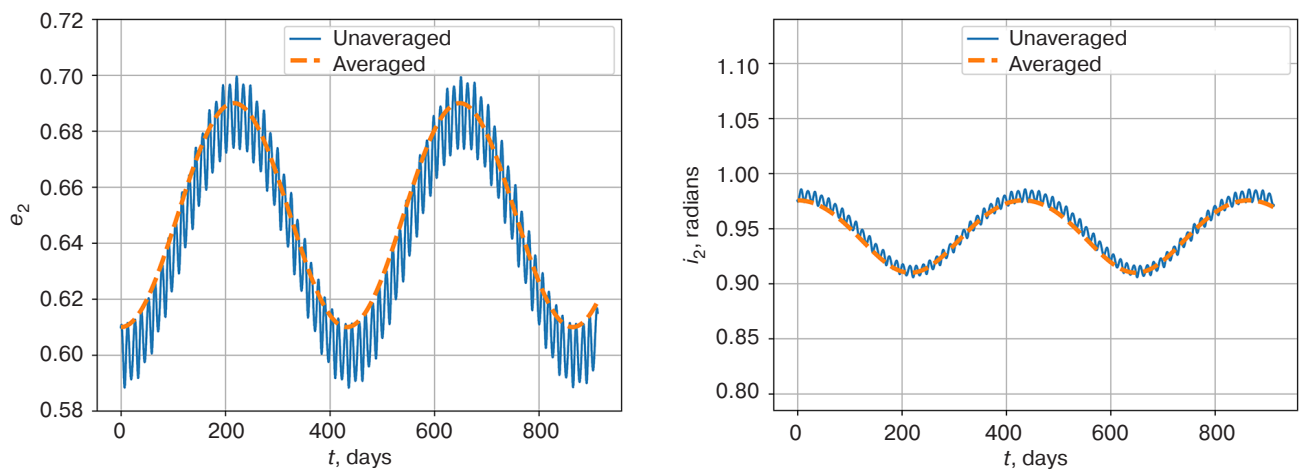


Fig. 9. Integral curves $e_2(t)$ and $i_2(t)$ of averaged and unaveraged systems of equations of satellite orbital motion

The graphs obtained also show periodic behavior and the main period, as in the averaged model, but with high-frequency oscillations superimposed upon it. The total amplitude of the oscillations corresponds to the amplitude of the averaged line, while the instantaneous values of eccentricity and inclination deviate from the averaged value. The averaged system is a “smoothed” version of the unaveraged system. The integral curves of the unaveraged system of equations contain more details and information about short-term changes in orbital elements. The averaged system can be useful for long-term forecasting and analysis of general trends. The unaveraged system is necessary for accurate modeling and prediction of satellite behavior over short time intervals. Although more difficult to analyze, the unaveraged system provides a more complete picture of satellite behavior.

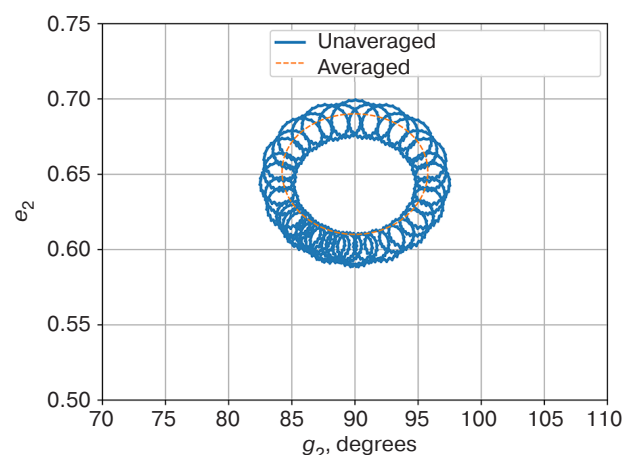


Fig. 10. Integral curves of averaged and unaveraged systems of equations of satellite orbital motion in a plane (g_2, e_2)

⁴ <https://ssd.jpl.nasa.gov/horizons/#api>. Accessed February 04, 2025.

⁵ Lunar Reconnaissance Orbiter is a robotic spacecraft orbiting the Moon. Launched on June 18, 2009. <https://www.n2yo.com/satellite/?s=35315>. Accessed February 04, 2025.

CONCLUSIONS

As a result of the study, averaged and unaveraged systems of equations of motion for the ASM were derived, taking into account the perturbation caused by the Earth's gravitational field, based on the system of equations of motion of the satellite in Delaunay canonical variables, enabling to track the change in its orbital parameters over time.

Stationary solutions of the averaged system of equations were found, and their stability was investigated. Conditions for the existence of "frozen" orbits were obtained, in which changes in inclination, eccentricity, and longitude of the pericenter from the ascending node are minimized. Phase portraits were constructed for the averaged system of equations at various values of the problem parameter which depends on the initial values of the inclination and eccentricity of the orbit. Integral curves of the non-averaged system of equations were also constructed.

It should be noted that the study takes into account the influence of the main factor on the satellite's orbit—external gravitational disturbance. Further study of this issue may enable a more accurate description of the motion of satellites around celestial bodies. The study has practical significance for the development and planning of space missions related to the exploration of the Moon and its vicinity. The equations obtained can be used to more accurately predict the orbital motion of satellites and determine the optimal parameters of their orbits.

ACKNOWLEDGMENTS

The work was carried out as part of the initiative research work of the Department of Higher Mathematics of the Institute of Artificial Intelligence at RTU MIREA (No. 192-III(VM)).

Authors' contribution

All authors contributed equally to the research work.

REFERENCES

1. Woodard M., Folta D.C., Woodfork D.W. ARTEMIS The First Mission to the Lunar Libration Orbits. In: *Conference: International Symposium on Space Flight Dynamics*. 2009. Available from URL: <https://www.researchgate.net/publication/235990349>. Accessed February 04, 2025.
2. Li C., Hu H., Yang M.-F., et al. Characteristics of the lunar samples returned by the Chang'E-5 mission. *Natl. Sci. Rev.* 2022;9(2):nwab188. <https://doi.org/10.1093/nsr/nwab188>
3. Li C., Hu H., Yang M.-F., et al. Nature of the lunar far-side samples returned by the Chang'E-6 mission. *Natl. Sci. Rev.* 2024;11(11):nwae328. <https://doi.org/10.1093/nsr/nwae328>
4. Mathavaraj S., Negi K. Chandrayaan-3 Trajectory Design: Injection to Successful Landing. *J. Spacecraft Rockets.* 2025;62(1):159–166. <https://doi.org/10.2514/1.A35980>
5. Kanu N.J., Gupta E., Verma N.J. An insight into India's Moon mission – Chandrayan-3: The first nation to land on the southernmost polar region of the Moon. *Planet. Space Sci.* 2024;242(5):105864. <https://doi.org/10.1016/j.pss.2024.105864>
6. Zelenyi L.M., Mitrofanov I.G., Tret'yakov V.I., Litvak M.L., Kalashnikov D.V., Surov A.V., Prokhorov V.G. Scientific program for the study of the spacecraft "Luna-25". In: *Automatic Spacecraft of the New Generation "Luna-25" – from Research to the Development of Lunar Resources*: in 2 v. Khimki; 2023. P. 8–28 (in Russ.). <https://elibrary.ru/lggmqz>
7. Lidov M.L. The evolution of orbits of artificial satellites of planets under the action of gravitational perturbations of external bodies. *Planet. Space Sci.* 1962;9(10):719–759. [https://doi.org/10.1016/0032-0633\(62\)90129-0](https://doi.org/10.1016/0032-0633(62)90129-0)
[Original Russian Text: Lidov M.L. The evolution of orbits of artificial satellites of planets under the action of gravitational perturbations of external bodies. *Iskusstvennye Sputniki Zemli*. 1961;8:5–45 (in Russ.)]
8. Kozai Y. Secular perturbations of asteroids with high inclination and eccentricity. *The Astronomical Journal*. 1962;67(9):591–598.
9. Vashkov'yak M.A., Teslenko N.M. Refined model for the evolution of distant satellite orbits. *Astron. Lett.* 2009;35(12):850–865. <https://doi.org/10.1134/S1063773709120056>
[Original Russian Text: Vashkov'yak M.A., Teslenko N.M. Refined model for the evolution of distant satellite orbits. *Pis'ma v Astronomicheskii zhurnal*. 2009;35(12):934–950 (in Russ.). <https://elibrary.ru/kygid>]
10. Vashkov'yak M.A. Constructive analytical solution of the evolution hill problem. *Sol. Syst. Res.* 2010;44(6):527–540. <https://doi.org/10.1134/S0038094610060067>
[Original Russian Text: Vashkov'yak M.A. Constructive analytical solution of the evolution hill problem. *Astronomicheskii Vestnik*. 2010;44(6):560–573 (in Russ.). <https://elibrary.ru/nbsuhf>]
11. Lidov M.L. On the approximate analysis of the evolution of artificial satellite orbits. In: *Problemy dvizheniya iskusstvennykh nebesnykh tel (Problems of the Motion of Artificial Celestial Bodies)*. Moscow: USSR Academy of Sciences; 1963. P. 119–134 (in Russ.).
12. Ely T.A. Stable Constellations of Frozen Elliptical Inclined Lunar Orbits. *J. Astronaut. Sci.* 2005;53(3):301–316. <https://doi.org/10.1007/BF03546355>

13. Goossens S., Sabaka T.J., Wicczorek M.A., Neumann G.A., Mazarico E., Lemoine F.G., et al. High-resolution gravity field models from GRAIL data and implications for models of the density structure of the Moon's crust. *Journal of Geophysical Research: Planets (JGR Planets)*. 2020;125(2):e2019JE006086. <https://doi.org/10.1029/2019JE006086>
14. Folta D.C., Pavlak T.A., Haapala A.F., Howell K.C., Woodard M.A. Earth–Moon libration point orbit stationkeeping: Theory, modeling, and operations. *Acta Astronautica*. 2013;94(1):421–433. <https://doi.org/10.1016/j.actaastro.2013.01.022>
15. Jadala G., Meedinti G.N., Delhibabu R. Satellite Orbit Prediction Using a Machine Learning Approach. *ICAI Workshops*. 2022. P. 28–46.
16. Ovchinnikov M., Shirobokov M., Trofimov S. Lunar Satellite Constellations in Frozen Low Orbits. *Aerospace*. 2024;11(11):918. <https://doi.org/10.3390/aerospace11110918>
17. Aksenov E.P. *Spetsial'nye funktsii v nebesnoi mekhanike (Special Functions in Celestial Mechanics)*. Moscow: Nauka; 1986, 320 p. (In Russ.).
18. Duboshin G.N. *Nebesnaya mekhanika. Osnovnye zadachi i metody (Celestial Mechanics. Basic Problems and Methods)*. Moscow: Nauka; 1975, 800 p. (In Russ.).
19. Murray C., Dermott S. *Dinamika Solnechnoi sistemy (Solar System Dynamics)*; transl. from Engl. Moscow: Fizmatlit; 2010, 588 p. (In Russ.). ISBN 978-5-9221-1121-8
[Murray C.D., Dermott S.F. *Solar System Dynamics*. Cambridge University Press; 1999, 592 p.]
20. Vil'ke V.G. *Mekhanika sistem material'nykh toчек i tverdykh tel (Mechanics of Systems of Material Points and Rigid Bodies)*. Moscow: Fizmatlit; 2013, 268 p. (In Russ.). ISBN 978-5-9221-1481-3

СПИСОК ЛИТЕРАТУРЫ

1. Woodard M., Folta D.C., Woodfork D.W. ARTEMIS The First Mission to the Lunar Libration Orbits. In: *Conference: International Symposium on Space Flight Dynamics*. 2009. URL: <https://www.researchgate.net/publication/235990349>. Дата обращения 04.02.2025. / Accessed February 04, 2025.
2. Li C., Hu H., Yang M.-F., et al. Characteristics of the lunar samples returned by the Chang'E-5 mission. *Natl. Sci. Rev.* 2022;9(2):nwab188. <https://doi.org/10.1093/nsr/nwab188>
3. Li C., Hu H., Yang M.-F., et al. Nature of the lunar far-side samples returned by the Chang'E-6 mission. *Natl. Sci. Rev.* 2024;11(11):nwae328. <https://doi.org/10.1093/nsr/nwae328>
4. Mathavaraj S., Negi K. Chandrayaan-3 Trajectory Design: Injection to Successful Landing. *J. Spacecraft Rockets*. 2025;62(1):159–166. <https://doi.org/10.2514/1.A35980>
5. Kanu N.J., Gupta E., Verma N.J. An insight into India's Moon mission – Chandrayan-3: The first nation to land on the southernmost polar region of the Moon. *Planet. Space Sci.* 2024;242(5):105864. <https://doi.org/10.1016/j.pss.2024.105864>
6. Зеленый Л.М., Митрофанов И.Г., Третьяков В.И., Литвак М.Л., Калашников Д.В., Суков А.В., Прохоров В.Г. Научная программа исследований космического аппарата «Луна-25». В кн.: *Автоматический космический аппарат нового поколения «Луна-25» – от исследования к освоению лунных ресурсов: в 2 т.* Химки: Научно-производственное объединение им. С.А. Лавочкина; 2023. С. 8–28. <https://elibrary.ru/lggmqz>
7. Лидов М.Л. Эволюция орбит искусственных спутников планет под действием гравитационных возмущений внешних тел. *Искусственные спутники Земли*. 1961;8:5–45.
8. Kozai Y. Secular perturbations of asteroids with high inclination and eccentricity. *The Astronomical Journal*. 1962;67(9):591–598.
9. Вашковьяк М.А., Тесленко Н.М. Уточненная модель эволюции далеких спутниковых орбит. *Письма в Астрономический журнал*. 2009;35(12):934–950. <https://elibrary.ru/kygisd>
10. Вашковьяк М.А. Конструктивно-аналитическое решение эволюционной задачи Хилла. *Астрономический вестник*. 2010;44(6):560–573. <https://elibrary.ru/nbsuhf>
11. Лидов М.Л. О приближенном анализе эволюции орбит искусственных спутников. В кн.: *Проблемы движения искусственных небесных тел*. М.: Изд-во АН СССР; 1963. С. 119–134.
12. Ely T.A. Stable Constellations of Frozen Elliptical Inclined Lunar Orbits. *J. Astronaut. Sci.* 2005;53(3):301–316. <https://doi.org/10.1007/BF03546355>
13. Goossens S., Sabaka T.J., Wicczorek M.A., Neumann G.A., Mazarico E., Lemoine F.G., et al. High-resolution gravity field models from GRAIL data and implications for models of the density structure of the Moon's crust. *Journal of Geophysical Research: Planets (JGR Planets)*. 2020;125(2):e2019JE006086. <https://doi.org/10.1029/2019JE006086>
14. Folta D.C., Pavlak T.A., Haapala A.F., Howell K.C., Woodard M.A. Earth–Moon libration point orbit stationkeeping: Theory, modeling, and operations. *Acta Astronautica*. 2013;94(1):421–433. <https://doi.org/10.1016/j.actaastro.2013.01.022>
15. Jadala G., Meedinti G.N., Delhibabu R. Satellite Orbit Prediction Using a Machine Learning Approach. *ICAI Workshops*. 2022. P. 28–46.
16. Ovchinnikov M., Shirobokov M., Trofimov S. Lunar Satellite Constellations in Frozen Low Orbits. *Aerospace*. 2024;11(11):918. <https://doi.org/10.3390/aerospace11110918>
17. Аksenov E.P. *Специальные функции в небесной механике*. М.: Наука; 1986, 320 с.
18. Дубошин Г.Н. *Небесная механика. Основные задачи и методы*. М.: Наука; 1975, 800 с.
19. Мюррей К., Дермотт С. *Динамика Солнечной системы: пер. с англ.* М.: Физматлит; 2010, 588 с. ISBN 978-5-9221-1121-8
20. Вильке В.Г. *Механика систем материальных точек и твердых тел*. М.: Физматлит; 2013, 268 с. ISBN 978-5-9221-1481-3

About the Authors

Olga V. Meshkova, Master Student, Department of Higher Mathematics, Institute of Artificial Intelligence, MIREA – Russian Technological University (78, Vernadskogo pr., Moscow, 119454 Russia). E-mail: oxn.lar5@yandex.ru. <https://orcid.org/0009-0002-2917-1025>

Albina V. Shatina, Dr. Sci. (Phys.-Math.), Docent, Head of the Department of Higher Mathematics, Institute of Artificial Intelligence, MIREA – Russian Technological University (78, Vernadskogo pr., Moscow, 119454 Russia). E-mail: shatina_av@mail.ru. Scopus Author ID 6506958326, RSCI SPIN-code 8714-6450, <https://orcid.org/0000-0001-5016-5899>

Об авторах

Мешкова Ольга Вячеславовна, магистрант, кафедра высшей математики, Институт искусственного интеллекта, ФГБОУ ВО «МИРЭА – Российский технологический университет» (119454, Россия, Москва, пр-т Вернадского, д. 78). E-mail: oxn.lar5@yandex.ru . <https://orcid.org/0009-0002-2917-1025>

Шатина Альбина Викторовна, д.ф.-м.н., доцент, заведующая кафедрой высшей математики, Институт искусственного интеллекта, ФГБОУ ВО «МИРЭА – Российский технологический университет» (119454, Россия, Москва, пр-т Вернадского, д. 78). E-mail: shatina_av@mail.ru. Scopus Author ID 6506958326, SPIN-код РИНЦ 8714-6450, <https://orcid.org/0000-0001-5016-5899>

*Translated from Russian into English by Lyudmila O. Bychkova
Edited for English language and spelling by Dr. David Mossop*