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RESEARCH ARTICLE

Modeling of surface waves in photonic crystal structures with a refractive index profile decreasing with distance from the surface

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Abstract

Objectives. Identification of the propagation patterns of surface waves in inhomogeneous and nonlinear crystal structures using mathematical models is an important fundamental problem in condensed matter physics, specifically waveguide optics. Models of waveguide structures used to establish an exact analytical solution are of particular significance. The aim of this work is to carry out a theoretical study of transversely polarized surface electric waves propagating along a photonic crystal with a certain refractive index profile.

Methods. The methods of mathematical physics, analysis, differential equations, and theory of special functions, as well as physical models of waveguide optics, were used in this study.

Results. A generalized hyperbolic permittivity profile was proposed to describe the spatially inhomogeneous distribution of the optical properties of a photonic crystal. This profile has a wide range of possibilities for varying its shape, allowing it to be used for a wide range of problems not limited to waveguide optics. An exact analytical solution of the wave equation with the selected permittivity profile was found in terms of the Whittaker function. Frequent cases of the generalized profile for which exact analytical solutions were indicated were also considered. These are expressed through the Whittaker and Macdonald functions. The study also describes surface transverse electric waves, where the field is localized near the surface of the photonic crystal and decreases with distance from it. The solution obtained also describes waveguide modes in which the field decreases with distance from the surface of the photonic crystal with oscillations. New features of surface wave localization were established. These were caused by a change in the parameters of the generalized hyperbolic profile modeling the dependence of the permittivity. It was also established that the maximum intensity of the surface wave is located in the photonic crystal.

Conclusions. The results of the description of the characteristics of surface waves obtained expand the theoretical concepts of waveguide optics. They can be useful in predicting the optical properties of various photonic crystal structures, as well as in designing various waveguide structures with the required dispersion-optical characteristics.

Keywords: inhomogeneous optical media, photonic crystal, surface waves, guided waves, waveguide modes

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НАУЧНАЯ СТАТЬЯ

Моделирование поверхностных волн в фотонных кристаллических структурах с профилем показателя преломления, убывающим с расстоянием от поверхности

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Резюме
Цели. Выявление закономерностей распространения поверхностных волн в неоднородных и нелинейных кристаллических структурах на основе математических моделей является важной фундаментальной задачей в физике конденсированного состояния, относящейся к волноводной оптике. При этом особой важностью обладают такие модели волноводных структур, в которых удается найти точное аналитическое решение. Цель работы – теоретическое изучение поверхностных электрических волн поперечной поляризации, распространяющихся вдоль фотонного кристалла с определенной формой профиля показателя преломления.
Методы. Применены методы математической физики, анализа, дифференциальных уравнений и теории специальных функций, а также физические модели волноводной оптики.
Результаты. Для описания пространственно неоднородного распределения оптических свойств фотонного кристалла предложен обобщенный гиперболический профиль диэлектрической проницаемости, который обладает широкими возможностями вариации формы, что позволяет его использовать для широкого круга задач, не ограничиваясь волноводной оптикой. Найдено точное аналитическое решение волнового уравнения для выбранного профиля диэлектрической проницаемости, выражаемое через функцию Уиттекера. Рассмотрены частые случаи обобщенного профиля, для которого указаны точные аналитические решения, выражаемые через функции Уиттекера и Макдональда. Описаны поверхностные поперечные электрические волны, поле в которых локализовано вблизи поверхности фотонного кристалла и убывает при удалении от нее. Полученное решение также описывает волноводные моды, в которых поле убывает при удалении от поверхности фотонного кристалла с осцилляциями. Выявлены новые особенности локализации поверхностных волн, обусловленные изменением параметров обобщенного гиперболического профиля, моделирующего зависимость диэлектрической проницаемости. Установлено, что в линейной поверхностной волне максимум интенсивности расположен в фотонном кристалле.
Выводы. Полученные результаты описания характеристик поверхностных волн расширяют теоретические представления волноводной оптики. Они могут быть полезны для прогнозирования оптических свойств различных фотонных кристаллических структур, а также при проектировании различных волноводных структур с требуемыми дисперсионно-оптическими характеристиками.

Ключевые слова: неоднородные оптические среды, фотонный кристалл, поверхностные волны, управляемые волны, волноводные моды

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INTRODUCTION

The study of the properties of surface waves in crystals, including photonic heterostructures, by mathematical modeling is an important fundamental problem [1]. Of special significance are such models of waveguide structures which can be used to find an exact analytical solution.

When describing real crystals,the refractive index (or permittivity) profile which models the optical inhomogeneity is selected, in such a way that it best fits the experimental data. For example [2], the photonic band gap in one-dimensional exponentially graded photonic crystals was investigated using an exponential profile, in order to represent the change in refractive index between the initial and final boundaries of the graded layer. An exponential refractive index profile was also used to describe the properties of a new photonic crystal resonator structure [3]. A simple hyperbolic refractive index profile in the form $n \sim 1/x$, characterizing its decrease with distance x from the contact surface, was used in the description of photonic band gaps [4, 5]. It was also used in a new approach to achieve improved sensitivity characteristics of photonic crystal [6].

The problem of finding exact solutions to the wave equation in nonlinear and inhomogeneous media is associated with the choice of the refractive index (permittivity) profile in inhomogeneous media [7]. Many exactly integrable refractive index profiles are known [8–11]. However, there remain a significant number of problems involving the definition of explicit analytical expressions for surface waves. These are described not by solutions of individual equations, but by solutions of boundary-value problems of systems of equations in certain domains which satisfy the conjugation conditions at their boundaries [12–14].

This paper presents the results of a theoretical study of surface waves, or more precisely, transversely polarized electric waves propagating along a photonic crystal with a specific refractive index profile. A new exact analytical solution was obtained for this spatial dependence, which is a generalization of the hyperbolic profile [15]. It was used to describe new types of linear and nonlinear surface waves. Linear surface waves for a simple hyperbolic permittivity profile $\varepsilon \sim 1/x$ were recently described [16]. Nonlinear surface waves propagating along the contact of an inhomogeneous medium with such a profile with a nonlinear medium with step nonlinearity were also described [17]. In this work, surface waves with step nonlinearity were studied using a generalized hyperbolic permittivity profile. Such a profile has wide possibilities for varying its shape which allows its use for a wide range of problems not limited to waveguide optics [18].

1. MODEL OF A PHOTONIC CRYSTAL STRUCTURE WITH A VARIABLE REFRACTIVE INDEX

Let us consider a photonic crystal in which the refractive index decreases with distance from its surface. Such a decreasing profile can be achieved by injecting ions of specially selected impurities [5]. Ion implantation technologies are well developed and allow the formation of an ion distribution which will provide the desired spatial refractive index profile [2–6]. This paper considers models of hetero-structures made of nonmagnetic materials consisting of a photonic crystal in contact with a medium with uniform optical characteristics.

Let the Ox axis be perpendicular to the surface of the photonic crystal which is assumed to be flat. Then the crystal surface coincides with the plane $x = 0$. The Oy and Oz axes are located in this plane. The spatial distributions of the optical properties of the photonic heterostructure under consideration over the surface are assumed to be uniform. They change only in the direction transverse to the surface.

Let us consider only transverse monochromatic waves propagating along the surface of the crystal with the electric field strength component:

$$E_y(x,z) = u(x)e^{i(knz-\omega t)}, \quad (1)$$

wherein $u(x)$ is the transverse profile of the electric field strength, $k = 2\pi/\lambda$ is the wave number, λ is the wavelength, $n = ck/\omega$ is the effective refractive index, ω is the frequency, and c is the speed of light in a vacuum.

The equation for finding the transverse profile of the electric field strength has the following form [7]:

$$u''(x) + \{\varepsilon(x, I) - n^2\}k^2 u(x) = 0. \quad (2)$$

Here, $\varepsilon(x, I)$ is the permittivity of the photonic hetero-structure which in general can reflect a nonlinear response in the form of a specific dependence on the electric field intensity $I = |u|^2$ (a model of nonlinearity of the medium). It also characterizes the optical inhomogeneity modeled by a dependence on the spatial coordinate (a spatial profile). The square of the permittivity determines the refractive index.

In the case under consideration, when the photonic crystal with the flat surface is in contact with the optically homogeneous medium with a nonlinear response (nonlinear medium) or without it (linear medium), the permittivity can be represented as follows [13]:

$$\varepsilon(x, I) = \begin{cases} \varepsilon_{\text{in}}(x), & x > 0, \\ \varepsilon_{\text{ho}}(I), & x < 0. \end{cases} \quad (3)$$

Here, $\varepsilon_{\text{in}}(x)$ models the inhomogeneity of the permittivity profile as a function of the coordinate in the direction perpendicular to the surface of the photonic crystal, and $\varepsilon_{\text{ho}}(I)$ models the nonlinear response of the optically homogeneous medium in contact with the photonic crystal.

If the transverse profile of the electric field strength is written as:

$$u(x) = \begin{cases} u_{\text{in}}(x), & x > 0 \\ u_{\text{ho}}(I), & x < 0, \end{cases} \quad (4)$$

wherein $u_{\text{in}}(x)$ and $u_{\text{ho}}(I)$ are the electric field strength profiles defined on the half-axes in the photonic crystal and in the optically homogeneous medium, respectively, then Eq. (2) is divided into two equations on the half-axes:

$$u''_{\text{in}}(x) + \{\varepsilon_{\text{in}}(x) - n^2\}k^2 u_{\text{in}}(x) = 0, \ x > 0, \quad (5)$$

$$u''_{\text{ho}}(x) + \{\varepsilon_{\text{ho}}(I) - n^2\}k^2 u_{\text{ho}}(x) = 0, \ x < 0. \quad (6)$$

The continuity of the field components on the surface of the photonic crystal determines the boundary conditions of conjugation at $x = 0$:

$$u_{\text{in}}(+0) = u_{\text{ho}}(-0), \quad (7)$$

$$u'_{\text{in}}(+0) = u'_{\text{ho}}(-0). \quad (8)$$

Finally, the following conditions should be added which follow from the requirement that the field disappears at infinity: $u_{\text{ho}}(x) \to 0$, $u_{\text{in}}(x) \to 0$ as $|x| \to \infty$.

As a result, a model is formulated as conjugation boundary-value problem (5)–(8). The continuous and everywhere bounded solutions can be used to describe the propagation of surface waves in photonic crystal structures.

2. MODELING OF A DESCENDING REFRACTIVE INDEX PROFILE OF A PHOTONIC CRYSTAL STRUCTURE

In order to find solutions in explicit form, a specific form of the permittivity profile $\varepsilon_{\text{in}}(x)$ and the model $\varepsilon_{\text{ho}}(I)$ of nonlinearity of the medium need to be selected. The inhomogeneity of the photonic crystal structure is often modeled using decreasing profiles [2–6]. They describe a decrease in the refractive index with increasing distance from the surface into the depth of the photonic structure. The following dependence is proposed:

$$\varepsilon_{\text{in}}(x) = e_0 + \frac{e_1}{x + h_1} + \frac{e_2}{(x + h_2)^2}, \quad (9)$$

wherein e_0, e_1, e_2, h_1, and h_2 are parameters of the spatial profile of the permittivity.

This form of the permittivity profile is a generalization of the hyperbolic profile considered earlier [15–17]. This can be obtained from expression (9) at $e_0 = e_2 = 0$.

Substitution of profile (9) into Eq. (5) leads to the following equation:

$$u''_{\text{in}}(x) + \left(e_0 + \frac{e_1}{x + h_1} + \frac{e_2}{(x + h_2)^2} - n^2\right)k^2 u_{\text{in}}(x) = 0. \quad (10)$$

Using the definition of the Heun confluent function $H_c(q, \alpha, \gamma, \delta, e, z)$ as a solution $y(z)$ of the Heun confluent differential equation [19]

$$z(z-1)y'' + (\gamma(z-1) + \delta z + z(z-1)e)y' + (\alpha z - q)y = 0,$$

the general solution of Eq. (10) can be written with its help in the following form:

$$u_{\text{in}}(x) = (x + h_1)e^{n_0 kx}\left\{C_1(x + h_2)^{\frac{n_1+1}{2}} H_c \times\right.$$

$$\times\left(2kn_0(h_2 - h_1), n_1, 1, k^2 e_1(h_2 - h_1), \frac{1}{2}, \frac{x + h_2}{h_2 - h_1}\right) +$$

$$+ C_2(x + h_2)^{-\frac{n_1+1}{2}} H_c \times$$

$$\left.\times\left(2kn_0(h_2 - h_1), -n_1, 1, k^2 e_1(h_2 - h_1), \frac{1}{2}, \frac{x + h_2}{h_2 - h_1}\right)\right\},$$

wherein $n_0 = \sqrt{n^2 - e_0}$, $n_1 = \frac{1}{2}\sqrt{1 - 4k^2 e_2}$, and C_1 and C_2 are integration constants.

The analysis of such a solution is clearly very difficult. However, if it is assumed that $h_1 = h_2 = h$, then, for the profile:

$$\varepsilon_{\text{in}}(x) = e_0 + \frac{e_1}{x+h} + \frac{e_2}{(x+h)^2}. \qquad (11)$$

Equation (10) has the following form:

$$u''_{\text{in}}(x) + \left(e_0 + \frac{e_1}{x+h} + \frac{e_2}{(x+h)^2} - n^2\right)k^2 u_{\text{in}}(x) = 0. \qquad (12)$$

Given this simplification, the general solution of Eq. (12) is expressed in terms of the Whittaker functions $W_{\mu,\nu}(z)$ and $M_{\mu,\nu}(z)$ as solutions $y(z)$ of the Whittaker differential equation [19]:

$$y'' + \left(\frac{\mu}{z} - \frac{1}{4} + \frac{1/4 - \nu^2}{z^2}\right)y = 0,$$

which is clearly simpler than the Heun differential equation.

Since a solution of the Whittaker equation which is bounded and decreases at infinity is the Whittaker function $W_{\mu,\nu}(z)$, then the transverse profile of the electric field strength in the photonic crystal structure determined by the solution of Eq. (10) can be written as follows:

$$u_{\text{in}}(x) = u_0 \frac{W_{\mu,\nu}(2n_0 k(x+h))}{W_{\mu,\nu}(2n_0 kh)}. \qquad (13)$$

Here, u_0 is the value of the electric field strength on the surface of the photonic crystal, and the parameters of the Whittaker function are determined by the parameters of Eq. (11) as:

$$\mu = e_1 k / 2n_0 = e_1 k / 2\sqrt{n^2 - e_0},$$

$$\nu = n_1 = \sqrt{1 - 4k^2 e_2} / 2.$$

Hence, it follows that solution (13) exists under the conditions $n^2 > e_0$ and $k^2 > 1/4e_2$.

Thus, the spatial distribution of the electric field strength in the photonic crystal structure with the permittivity profile decreasing with distance from the surface according to generalized hyperbolic law (11) was obtained in explicit analytical form (12).

Note that the use of the generalized profile (11) allows the solution obtained (13) to be applied not only in the theory of waveguide optics, but also in quantum mechanics. Equation (12) can be considered as the stationary Schrödinger equation [18] with a generalized hyperbolic potential described by profile (11). Moreover, varying the values of the parameters of profile (11) enables significantly different forms of it to be obtained (Fig. 1). These are applicable in describing both monotonic (Fig. 1a) and nonmonotonic (Fig. 1b) potential barriers, as well as potential wells (Fig. 1c).

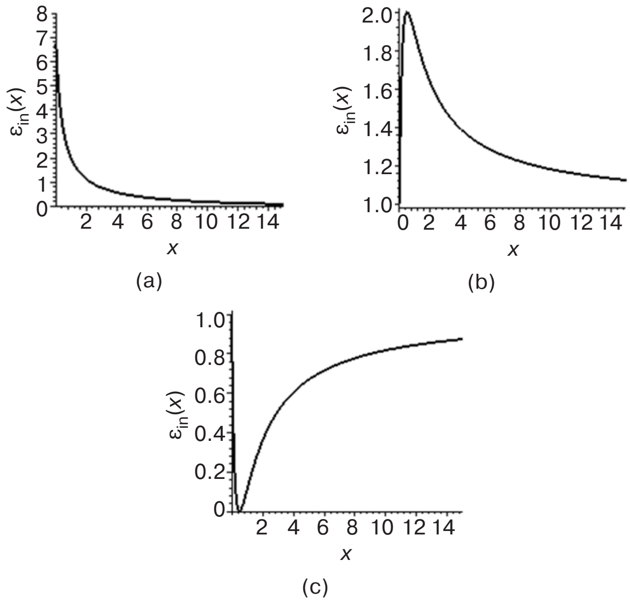

Fig. 1. Spatial profiles of permittivity (11) at various values of its parameters (in conventional dimensionless units): (a) $e_0 = -0.1$, $e_1 = 3$, and $e_2 = 0.2$; (b) $e_0 = -1$, $e_1 = 2$, and $e_2 = -1$; and (c) $e_0 = 1$, $e_1 = -2$, $e_2 = 1$, and $h = 0.5$

Let us consider special cases of profile (9).

1) $e_2 = 0$. In this case, profile (9) takes the form of a dependence decreasing according to a hyperbolic law:

$$\varepsilon_{\text{in}}(x) = e_0 + \frac{e_1}{x+h}, \qquad (14)$$

and Eq. (12) is written as:

$$u''_{\text{in}}(x) + \left(e_0 + \frac{e_1}{x+h} - n^2\right)k^2 u_{\text{in}}(x) = 0. \qquad (15)$$

The general solution of Eq. (5) is expressed in terms of the Whittaker functions:

$$u_{\text{in}}(x) = C_1 W_{\mu,1/2}(2n_0 k(x+h)) + + C_2 W_{\mu,1/2}(2n_0 k(x+h)), \qquad (16)$$

wherein the parameters of the Whittaker functions are $\mu = e_1 k / 2n_0 = e_1 k / 2\sqrt{n^2 - e_0}$ and $\nu = 1/2$.

Since the description of surface waves requires that the solution should be bounded at infinity: $u_{in}(x) \to 0$ as $|x| \to \infty$, for further application of solution (16), it is necessary to set $C_2 = 0$. In order to satisfy conjugation condition (7) at the interface of the crystals, $C_1 = u_0 / W_{\mu,1/2}(2n_0kh)$, may be chosen. In this case, solution (16), which can be used to solve the boundary-value conjugation problem, takes the following form:

$$u_{in}(x) = u_0 \frac{W_{\mu,1/2}(2n_0k(x+h))}{W_{\mu,1/2}(2n_0kh)}. \quad (17)$$

2) $e_1 = 0$. In this case, profile (9) takes the form of a dependence decreasing according to a hyperbolic law:

$$\varepsilon_{in}(x) = e_0 + \frac{e_2}{(x+h)^2}, \quad (18)$$

and Eq. (12) is represented as:

$$u''_{in}(x) + \left(e_0 + \frac{e_2}{(x+h)^2} - n^2\right)k^2 u_{in}(x) = 0. \quad (19)$$

The study found a general solution to Eq. (19):

$$u_{in}(x) = \sqrt{x+h} \times \times \{C_1 I_\nu(2n_0k(x+h)) + C_2 K_\nu(2n_0k(x+h))\}. \quad (20)$$

Wherein $I_\nu(z)$ and $K_\nu(z)$ are modified cylindrical functions of the imaginary argument of the first and second kind, also called the Infeld and Macdonald functions, respectively, of the order $\nu = n_1 = \sqrt{1-4k^2e_2}/2$. These functions are linearly independent solutions of the modified Bessel differential equation [19]:

$$y'' + \frac{1}{z}y' - \left(1 + \frac{\nu^2}{z^2}\right)y = 0,$$

which is obviously simpler in comparison with both the Heun differential equation and the Whittaker differential equation.

Since the Infeld function is unbounded, $C_1 = 0$ needs to be set for the further application of solution (20). In order to satisfy conjugation condition (7) at the interface of the crystals, $C_2 = u_0\sqrt{h} / K_\nu(2n_0kh)$, can be chosen. In this case the solution (20), which can be used to solve the boundary-value conjugation problem, takes the following form:

$$u_{in}(x) = u_0\sqrt{1+\frac{x}{h}}\frac{K_\nu(2n_0k(x+h))}{K_\nu(2n_0kh)}. \quad (21)$$

It should be noted that solution (17) is such a special case of solution (13) obtained at $e_2 = 0$ in an obvious way, while solution (21) does not follow explicitly from solution (13) at $e_1 = 0$. However, its form is preferable for use in constructing a solution to the boundary-value conjugation problem in modeling surface waves. The Macdonald function contains one parameter less than the Whittaker function, which simplifies the analysis of the solution.

3. SURFACE WAVES ALONG A PHOTONIC CRYSTAL WITH A DECREASING REFRACTIVE INDEX

In the case under consideration, when the photonic crystal is in contact with air or with an optically homogeneous dielectric without a nonlinear response (linear medium), the permittivity in this half-space is assumed to be constant and independent of the field intensity: $\varepsilon_{ho}(I) = \varepsilon_0$.

As a result, the photonic crystal structure under consideration has a permittivity of:

$$\varepsilon(x,I) = \begin{cases} e_0 + \dfrac{e_1}{x+h} + \dfrac{e_2}{(x+h)^2}, & x > 0, \\ \varepsilon_0, & x < 0. \end{cases} \quad (22)$$

Then, taking into account the assumptions made, Eq. (6) describing the distribution of the field in a homogeneous medium is written as follows:

$$u''_{ho}(x) - q_0^2 u_{ho}(x) = 0, \ x < 0, \quad (23)$$

wherein $q_0^2 = k^2(n^2 - \varepsilon_0)$.

The solution of Eq. (15) decreasing at infinity has the following form:

$$u_{ho}(x) = u_0 e^{q_0 x}. \quad (24)$$

The surface wave is described by the solution of boundary-value conjugation problem (5)–(8). To find this solution, the parameters of obtained solutions (13) and (24) on the semi-axes need to be defined in such a way that they satisfy boundary conditions (7) and (8). Solutions (13) and (24) clearly satisfy continuity condition (7) on the surface of the photonic crystal. In order to satisfy the continuity condition of the derivatives, i.e., to ensure the smoothness of the transverse profile of the surface wave, solutions (13) and (24) are substituted into boundary condition (8). This gives the following dispersion equation:

$$q_0 = \frac{k}{n_0}(n_0^2 - e_1/2) - \frac{W_{\mu+1,\nu}(2n_0kh)}{hW_{\mu,\nu}(2n_0kh)}. \quad (25)$$

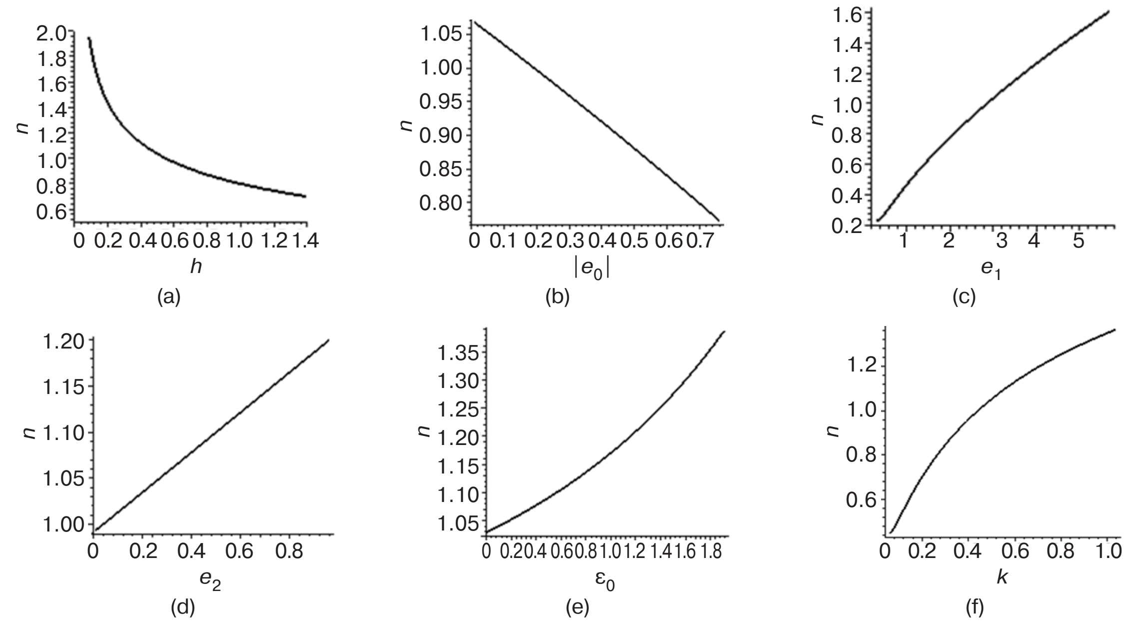

Fig. 2. Dependencies of the effective refractive index (in conventional dimensionless units):
(a) on h at $e_0 = -0.1$, $e_1 = 3$, $e_2 = 0.2$, $\mathrm{e}_0 = 0.05$, and $k = 0.5$;
(b) on e_0 at $h = 0.5$, $e_1 = 3$, $e_2 = 0.2$, $\mathrm{e}_0 = 0.05$, and $k = 0.5$;
(c) on e_1 at $h = 0.5$, $e_0 = -0.1$, $e_2 = 0.2$, $\mathrm{e}_0 = 0.05$, and $k = 0.5$;
(d) on e_2 at $h = 0.5$, $e_0 = -0.1$, $e_1 = 3$, $\mathrm{e}_0 = 0.05$, and $k = 0.5$;
(e) on e_0 at $h = 0.5$, $e_0 = -0.1$, $e_1 = 3$, $e_2 = 0.2$, and $k = 0.5$;
and (f) on k at $h = 0.5$, $e_0 = -0.1$, $e_1 = 3$, $e_2 = 0.2$, and $\mathrm{e}_0 = 0.05$

The dispersion equation defines a continuous spectrum of values of the effective refractive index n as a function of the parameters of permittivity profile (22). Figure 2 presents the results of the numerical solution of dispersion Eq. (25). The effective refractive index increases with increasing parameters e_0, e_1, e_2, ε_0, and k and decreases only with increasing h.

Thus, the solution to boundary-value problem (5)–(8), describing a surface wave propagating along the surface of the photonic crystal, is obtained after substituting solutions (13) and (24) into the following expressions (4):

$$u(x) = u_0 \begin{cases} \dfrac{W_{\mu,\nu}(2n_0 k(x+h))}{W_{\mu,\nu}(2n_0 kh)}, & x > 0, \\ e^{q_0 x}, & x < 0. \end{cases} \quad (26)$$

Figure 3 presents the transverse profiles of the electric field strength in surface wave (26).

The electric field is clearly localized in narrow regions near the surface on both sides, with the maximum intensity in the photonic crystal. The intensity can be higher in the photonic crystal than in the homogeneous dielectric, despite the fact that the field penetration depth in the photonic crystal may be less than that in the dielectric.

An increase in parameter h leads to an increase in the field localization width. The intensity maximum shifts into the depth of the photonic crystal, and its intensity increases (Fig. 3a). With a decrease in the parameter e_0 (an increase in its absolute value), the field penetration depth into the homogeneous dielectric increases and that into the photonic crystal decreases. The intensity maximum shifts toward the surface, and its intensity also decreases (Fig. 3b). An increase in the parameter e_1 leads to a decrease in the field localization width. The intensity of the maximum increases, and its position does not change (Fig. 3c). An increase in the parameter e_2 also leads to a decrease in the field localization width. However, the intensity of the maximum decreases, and its position shifts toward the surface (Fig. 3d). An increase in parameter ε_0 leads to almost the same effect, except that the field penetration depth into the homogeneous dielectric increases (Fig. 3d). An increase in the wavenumber (a decrease in the wavelength) leads to a decrease in the field localization width, the intensity of the maximum increases, and its position shifts slightly toward the surface (Fig. 3e).

The analysis shows that the field penetration depth into the photonic crystal decreases with increasing effective refractive index. Therefore, by adjusting the angle of incidence of the laser beam exciting the surface wave, the field penetration depth into the photonic crystal can be varied.

An important point is that the solution (26) to the formulated boundary-value problem (5)–(8) obtained describes not only a surface wave, in which the electric field strength profile decreases fairly rapidly with increasing distance from the surface, but also waveguide modes, in which the electric field strength profile decreases with oscillations. Waveguide modes are excited at specific parameter values. Their characteristic profiles are shown in Fig. 4. The oscillation amplitudes of waveguide modes decrease as they approach the crystal surface. The order of a waveguide mode can be defined as the number of maxima of the intensity $I = |u|^2$ (the number of maxima and minima of the electric field strength).

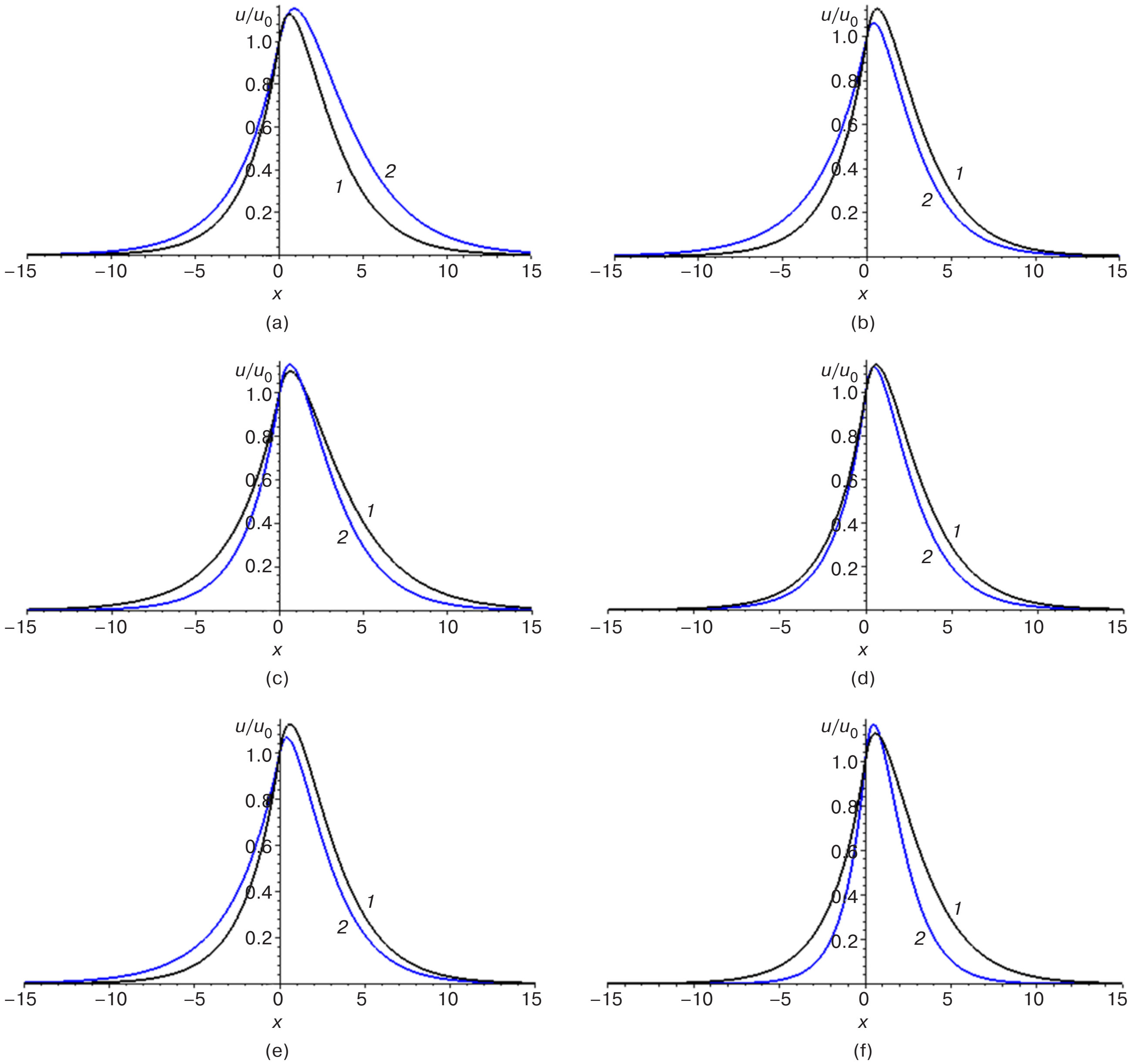

Fig. 3. Transverse profiles of the electric field strength in surface wave (26) at various values of the system parameters (in conventional dimensionless units):
(a) h = (*1*) 0.5 and (*2*) 0.9, $e_0 = -0.1$, $e_1 = 3$, $e_2 = 0.2$, $\mathrm{e}_0 = 0.05$, and $k = 0.5$;
(b) $h = 0.5$, e_0 = (*1*) −0.1 and (*2*) −0.9, $e_1 = 3$, $e_2 = 0.2$, $\mathrm{e}_0 = 0.05$, and $k = 0.5$;
(c) $h = 0.5$, $e_0 = -0.1$, e_1 = (*1*) 2 and (*2*) 3, $e_2 = 0.2$, $\mathrm{e}_0 = 0.05$, and $k = 0.5$;
(d) $h = 0.5$, $e_0 = -0.1$, $e_1 = 3$, e_2 = (*1*) 0.2 and (*2*) 0.9, $\mathrm{e}_0 = 0.05$, and $k = 0.5$;
(e) $h = 0.5$, $e_0 = -0.1$, $e_1 = 3$, $e_2 = 0.2$, e_0 = (*1*) 0.05 and (*2*) 0.5, and $k = 0.5$; and
(f) $h = 0.5$, $e_0 = -0.1$, $e_1 = 3$, $e_2 = 0.2$, $\mathrm{e}_0 = 0.05$, and k = (*1*) 0.5 and (*2*) 0.7

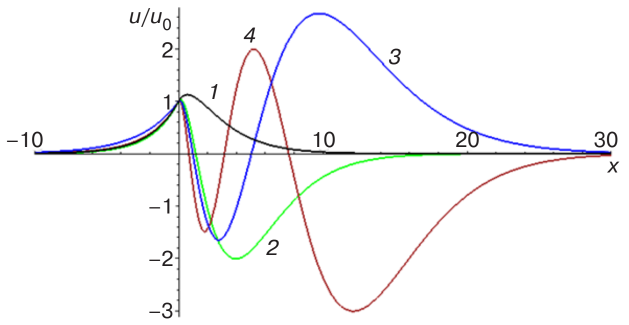

Fig. 4. Transverse profiles of the electric field strength of waveguide modes (26) at fixed values (in conventional dimensionless units) of $h = 0.5$, $e_0 = -0.1$, $e_2 = 0.2$, $e_0 = 0.05$, and $k = 0.5$ and various values of e_1 = (*1*) 3, (*2*) 9.1, (*3*) 10.1, and (*4*) 17.5

Let us now briefly consider a special case of profile (9) at $e_1 = 0$. Then the photonic crystal structure under consideration is characterized by a permittivity of:

$$\varepsilon(x, I) = \begin{cases} e_0 + \dfrac{e_2}{(x+h)^2}, x > 0, \\ \varepsilon_0, x < 0. \end{cases} \tag{27}$$

The mathematical description of the surface wave is now constructed from solutions (21) and (24). In order to find a solution to conjugation problem (5)–(8) in this case, the parameters of obtained solutions (21) and (24) on the semi-axes need to be defined, in such a way that they satisfy boundary conditions (7) and (8). Solutions (21) and (24) clearly satisfy continuity condition (7) on the surface of the photonic crystal. As a result, a solution to boundary-value conjugation problem (5)–(8) is obtained in the following form:

$$u(x) = u_0 \begin{cases} \sqrt{1 + \dfrac{x}{h}} \dfrac{K_\nu(2n_0 k(x+h))}{K_\nu(2n_0 kh)}, x > 0, \\ e^{q_0 x}, x < 0. \end{cases} \tag{28}$$

In a similar way, a dispersion equation is obtained:

$$\sqrt{\xi^2 - \varepsilon_0 (kh)^2} = \frac{1}{2} + \nu - \xi \frac{K_{\nu+1}(\xi)}{K_\nu(\xi)}, \tag{29}$$

wherein $\xi = n_0 kh$.

Dispersion Eq. (29) defines a continuous spectrum of values of the following effective refractive index:

$$n^2 = e_0 + (\xi / kh)^2, \tag{30}$$

wherein ξ is the positive root of Eq. (29).

A detailed analysis of solution (28) and the roots of the dispersion equation in the special case $e_0 = 0$ was given previously [20].

Thus, the exact analytical solutions of the boundary-value problem are obtained. In two cases using different functions, these solutions describe surface waves and waveguide modes propagating along a photonic crystal in contact with an optically homogeneous medium. The photonic crystal is characterized by the refractive index which decreases with distance from the contact surface in accordance with a generalized hyperbolic profile.

The formulated models and described characteristics of linear and nonlinear surface waves can be useful in predicting the optical properties of various photonic crystal structures, and multilayer composite optical structures [21, 22] used in optoelectronic engineering and photonics [23]. The results of this work can also be used in the design of various waveguide structures, including layered ones, with the required dispersion-optical characteristics.

CONCLUSIONS

This paper proposes models of photonic heterostructures with a spatial permittivity profile that allows for an exact analytical solution. A generalized hyperbolic profile was used to model this spatial distribution. Exact analytical solutions of the wave equation for the selected permittivity profiles were found. These are expressed in terms of the Whittaker and Macdonald functions.

A boundary-value problem was formulated to describe surface waves and waveguide modes propagating along the surface of the photonic crystal. This problem was solved using an exact solution for the generalized hyperbolic profile. Cases of contact between the photonic crystal and a homogeneous dielectric or a nonlinear optical medium were considered. Expressions were derived for surface transverse electric waves where the field is localized near the surface of the photonic crystal and decreases with distance from it.

A detailed analysis was made of the influence of the optical characteristics of the system for photonic crystal in contact with the homogeneous dielectric. This includes the parameters of the generalized hyperbolic permittivity profile, the permittivity of the homogeneous medium, and the wavenumber. A dispersion equation describing the dependence of the effective refractive index on the optical parameters of the system was derived and numerically analyzed. The study identified conditions for control parameters which allow for the localization of the electric field near the photonic crystal surface. It was shown that the solution to the boundary-value problem also describes waveguide modes, where the field decreases with oscillations with distance from the photonic crystal surface.

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