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RESEARCH ARTICLE

Image restoration using a discrete point spread function with consideration of finite pixel size

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Abstract
Objectives. The problem of restoring defocused and/or linearly blurred images using a Tikhonov-regularized inverse filter is considered. A common approach to this problem involves solving the Fredholm integral equation of the first convolution type by means of discretization based on quadrature formulas. The work sets out to obtain an expression of the point scattering function (PSF) taking into account pixel size finiteness and demonstrate its utility in application.
Methods. The research is based on signal theory and the method of digital image restoration using Tikhonov regularization.
Results. Taking into account the finiteness of the pixel size, discrete PSF formulas are obtained both for the case of a defocused image and for the case of a linearly blurred image at an arbitrary angle. It is shown that, while differences between the obtained formulas and those traditionally used are not significant under some conditions, under other conditions they can become significant.
Conclusions. In the case of restoring images at the resolution limit, i.e., when the pixel size cannot be considered negligibly small compared to the details of the image, the proposed approach can slightly improve the resolution. In addition, the derived formula for the discrete PSF corresponding to linear blur in an arbitrarily specified direction can be used to solve the problem without the need for prior image rotation and account for the blur value with sub-pixel accuracy. This offers an advantage in terms of improving the resolution of extremely fine details in the image, allowing the obtained formula to be used in solving the adaptive deconvolution problem, where precise adjustment of PSF parameters is required.

Keywords: blurred image, defocused image, resolution limit, finite pixel size, discrete PSF, image restoration, Tikhonov regularization, regularization parameter

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НАУЧНАЯ СТАТЬЯ

Восстановление изображений с использованием дискретной функции рассеяния точки, получаемой с учетом конечности размера пикселя

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Резюме
Цели. Рассматривается задача восстановления расфокусированного и/или линейно смазанного изображения с использованием регуляризированного по Тихонову инверсного фильтра. Распространенным подходом к решению этой задачи является решение интегрального уравнения Фредгольма 1-го рода типа свертки путем его дискретизации на основе квадратурных формул. Цель работы – получить выражение функции рассеяния точки (ФРТ) с учетом конечности размера пикселя и продемонстрировать его полезность.
Методы. Исследование основывается на теории сигналов и методе восстановления цифровых изображений с использованием тихоновской регуляризации.
Результаты. Получены формулы дискретной ФРТ как для случая расфокусированного, так и для случая линейно смазанного под произвольным углом изображения, с учетом конечности размера пикселя. Рассмотрены отличия полученных формул от традиционно используемых, показано при каких условиях эти отличия практически исчезают, а при каких – могут оказаться существенными.
Выводы. При восстановлении изображений на пределе разрешающей способности, т.е. когда размеры пикселя не могут считаться пренебрежимо малыми в сравнении с деталями изображения, предлагаемый подход может несколько улучшать разрешение. Кроме того, полученная формула дискретной ФРТ, соответствующей линейному смазу изображения в произвольно заданном направлении, позволяет не только решать задачу без необходимости предварительного поворота изображения, но и учитывать величину смаза с точностью до долей пикселя. Это дает преимущество в плане повышения разрешения предельно мелких деталей изображения и позволяет использовать данную формулу при решении задачи адаптивной деконволюции, когда требуется точная подстройка параметров ФРТ.

Ключевые слова: смазанное изображение, расфокусированное изображение, разрешающая способность, конечный размер пикселя, дискретная ФРТ, восстановление изображения, регуляризация по Тихонову, коэффициент регуляризации

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INTRODUCTION

In the contemporary world, the quality of images of various objects is critical in many fields. These include medical imaging, astronomy, earth remote sensing from satellites, security monitoring, and video surveillance. In order to meet the growing demand for high quality images, researchers are challenged to improve image reconstruction and processing techniques. One of the main challenges involves the recovery of images that have been distorted by uniform linear motion of the object or camera, leading to linear blur and defocus.

The present work continues the authors' earlier study [1] to explore the issue of restoring a linearly blurred or defocused image for a case where the blur parameters are known. So far, this problem has been the subject of many investigations. For example, the theory of solving inverse non-correlated problems, which includes the problem of image restoration, is the subject of fundamental works [2–5]. The image restoration problem is also specifically addressed in the fundamental works [6–10] published in the period leading up to the early 1990s. The state of the art in this field is described in [11–15]. However, all the above studies are based on point spread function (PSF) expressions that assume that the pixel size is infinitesimally small. By contrast, the present work derives PSF expressions that take pixel size finiteness into account, which offers several advantages. Firstly, considering the finiteness of the pixel size allows for some improvement in recovery quality when recovering images captured at the resolution limit of the camera, where the pixel size cannot be considered as negligibly small compared to the image details. This is true for both linearly blurred and defocused image reconstruction. In addition, the obtained PSF expressions are continuously dependent on the blur parameters, which allows easy adjustment of these parameters to the required values within fractions of a pixel. In particular, the value and direction of linear blur values can be easily selected.

The study aims to demonstrate the advantages of the proposed discrete PSF model that accounts for the finite pixel dimensions. The paper includes a rigorous mathematical derivation of the specified PSF equations and their comparison with traditional approaches. The theoretical results are confirmed by numerical simulation of the distortions under consideration and their elimination by deconvolution using the A.N. Tikhonov regularization.

1. THE 2D DISCRETE PSF WITH A LINEAR BLUR OF THE IMAGE IN AN ARBITRARY DIRECTION

We consider a rectangular panel of light-sensitive elements, which is an $M \times N$ pixel matrix. The pixels are assumed to be square-shaped and to fill the entire panel without gaps; let w be the pixel size. Each pixel is assigned a pair of indices (m, n), $m \in \overline{0, M-1}$; $n \in \overline{0, N-1}$, the pixel in the upper left corner of the panel having indices (0, 0). We relate this panel to the Cartesian coordinate system Oxy, with the origin in the upper left corner of the panel, such that the center of the pixel with indices (m, n) lies at the point with coordinates $(mw + w/2, nw + w/2)$. The Ox axis is vertically down, while the Oy axis is vertically to the right.

Let the function $p(x, y)$ define the luminance field of the points of the panel generated by the light flux forming the image at some instant of time t. The function $p(x, y)$ is logically independent of t. Then the luminance energy accumulated by the pixel with indices (m, n) for the exposure time τ of the image moving relative to the panel (focused flux) is equal to

$$q[m,n] = \\ = \int_{mw}^{(m+1)w} dx \int_{nw}^{(n+1)w} dy \int_0^{\tau} p(x - v_x(x,y)t,\ y - v_y(x,y)t)dt,$$

where $(v_x(x, y), v_y(x, y))$ are Cartesian components of the velocity vector of the image point with coordinates (x, y). So far, we have been considering the general case where different pixels can have different velocities.

The 2D Kotelnikov interpolation series can be used to represent the luminance field, as follows:

$$p(x,y) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} p[m,n] \operatorname{sinc}\left(\frac{x}{w} - m\right) \operatorname{sinc}\left(\frac{y}{w} - n\right), \quad (1)$$

where $p[m, n] = p(mw, nw)$.

Substituting this expression into the integral, we obtain the following:

$$q[k,l] = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} h_{k,l}[k-m, l-n] p[m,n],$$

where

$$h_{k,l}[m,n] = \\ = w^2 \int_0^{\tau} \operatorname{sinc}\left(m - \frac{v_x(kw,lw)t}{w}\right) \operatorname{sinc}\left(n - \frac{v_y(kw,lw)t}{w}\right) dt = \\ = w^2 \tau \int_0^1 \operatorname{sinc}\left(m - \frac{v_x(kw,lw)\tau}{w} t\right) \operatorname{sinc}\left(n - \frac{v_y(kw,lw)\tau}{w} t\right) dt.$$

Here, it is taken into account that the velocity field of the image motion within a pixel can be considered almost constant and equal to its value in the upper left corner of the pixel; in this case, the multiplier $w^2\tau$ is considered equal to one.

Under the assumption that the velocity field is constant over the entire pixel matrix, the 2D convolution is the following:

$$q[k,l]=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}h[m,n]p[k-m,l-n], \qquad (2)$$

where the kernel of this convolution is defined by the following equation:

$$h[m,n]=\int_0^1 \mathrm{sinc}(m-u_x t)\mathrm{sinc}(n-u_y t)dt, \qquad (3)$$

where $u_x = v_x\tau/w$, $u_y = v_y\tau/w$ are the displacement components in pixels for the exposure time.

With $u_x = u_y = 0$, we have $h[m, n] = \mathrm{sinc}(m)\mathrm{sinc}(n) = \delta[n]\delta[m]$, as it should be.

The examples of the graphs of the discrete kernel calculated by Eq. (3) are shown in Fig. 1.

In the general case, taking into account the finiteness of the pixel matrix size, we have a 2D finite convolution:

$$q[m,n]=\sum_{k=0}^{\min(m,\,K-1)}\sum_{l=0}^{\min(n,\,L-1)}h[k,l]p[m-k,n-l], \qquad (4)$$

where $m\in\overline{0,M-1}$; $n\in\overline{0,N-1}$ and array $p[:, :]$ is assumed to be of size $M \times N$; array $h[:, :]$ is of size $K \times L$; and array $q[:, :]$ is of size $(M + K) \times (N + L)$.

In particular, when $u_x = 0$ (no vertical displacement),

$$h[m,n]=\delta[m]\int_0^1 \mathrm{sinc}(n-u_y t)dt,$$

where $\delta[m]$ is a discrete delta function, i.e., in the absence of the vertical velocity component, the 2D convolution actually reduces to the 1D convolution with the kernel, as follows:

$$h[n]=\int_0^1 \mathrm{sinc}(n-u_y t)dt.$$

In this case, taking into account the finiteness of the pixel matrix size, we get

$$q[k]=\sum_{m=0}^{\min(k,M-1)}h[m]p[k-m],$$

where $k\in\overline{0,M-1}$.

If we add the multiplier $1/w^2$ to the right-hand side of Eq. (3) and then proceed to the limit at $w \to 0$, taking into account that the kernel does not depend on the integer indices m, n but on the corresponding continuous variables $x = mw$, $y = nw$, we obtain the equation of the following form:

$$h(x,y)=\int_0^1 \delta(x-v_x\tau t)\delta(y-v_y\tau t)dt.$$

Although this equation is used in some literature on optics (e.g., [16]), it is not suitable for direct discretization in this form. It can only be discretized by replacing the delta function it contains by a suitable regular function; such a replacement by the scaled sinc function leads back to Eq. (3). However, a slightly different transformation procedure is also possible to obtain an expression suitable for discretization:

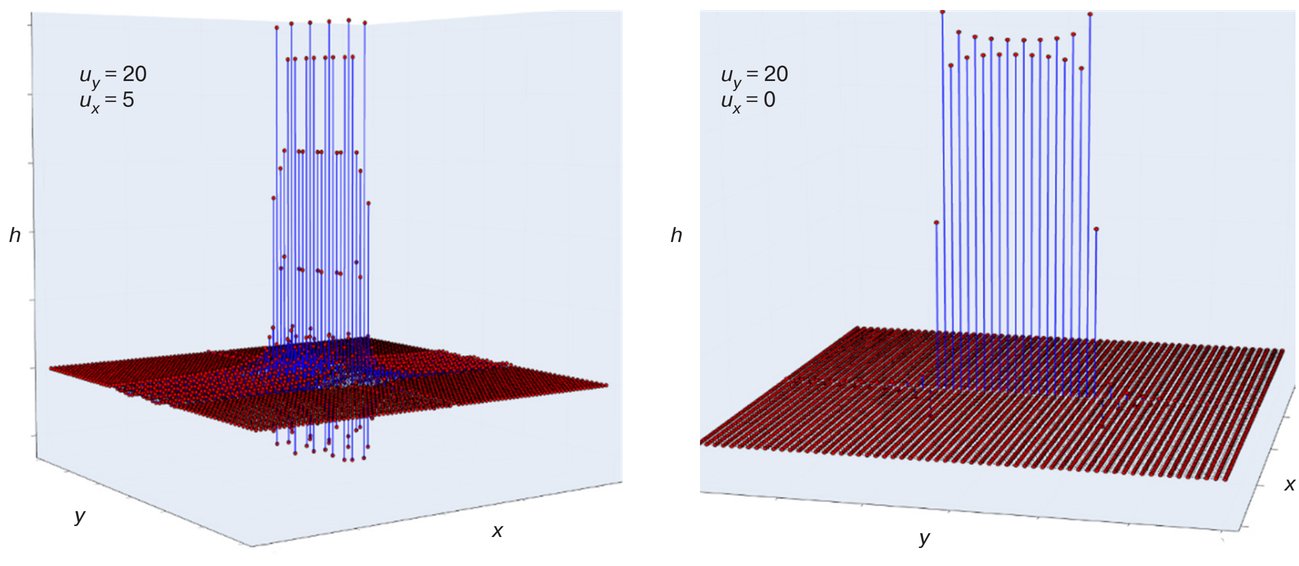


Fig. 1. Examples of graphs of the 2D discrete kernel of linear blur calculated by Eq. (3)

$$h(x,y)=\int_{-\infty}^{+\infty} I_{(0;1)}(t)\delta(x-v_x\tau t)\delta(y-v_y\tau t)dt=$$

$$=I_{(0;1)}\left(\frac{y}{v_y\tau}\right)\delta\left(x-\frac{v_x}{v_y}y\right),$$

where $I_{(0;1)}(y)$ is the indicator function of the interval (0; 1). Thus, given that the sinc function, when appropriately scaled, plays the role of the Dirac delta function in the space of functions with a finite frequency spectrum, we obtain

$$h(x,y)=I_{(0;v_y\tau)}(y)\,\mathrm{sinc}\left(\frac{1}{w}\left(x-\frac{v_x}{v_y}y\right)\right).$$

The scaling factor $1/w$ appearing in this substitution is discarded for convenience. Then, assuming again $x = mw$ and $y = nw$, we obtain the discrete analogue of the last equation, as follows:

$$h[m,n]=hu_y[n]\,\mathrm{sinc}\left(m-\frac{u_x}{u_y}n\right), \quad (5)$$

where $u_y=\frac{v_y\tau}{w}\in\mathbb{N}$, $hu_y[n]$ is the function of the integer argument at the extreme values of the argument $n = 0$, u_y equal to 1/2; at $n = 1, 2, ..., u_y - 1$ equal to 1, and at all other n equal to 0, that correspond to the quadrature formula of trapezoids.

It should be noted that the value of the horizontal blur u_y in Eq. (5) is assumed to be a positive integer, whereas this equation imposes no such restriction on the value of the vertical component of the blur u_x; u_x can take any real value in this equation.

In particular, if we set $u_x = 0$ in (5), then considering the identity $\mathrm{sinc}(m) = \delta[m]$, we get $h[m, n] = h_{u_y}[n]\delta[m]$. Since $\delta[m]$ only differs from zero when $m = 0$, the 2D kernel $h[m, n]$ is replaced by a 1D kernel, which is most commonly used in the literature to describe horizontal linear blur (e.g., [8, 14, 15]).

Equation (5) can be considered as an alternative to Eq. (3). Like Eq. (3), it allows the recovery of a linearly blurred image with arbitrary blur direction. However, in contrast to Eq. (5), Eq. (3) removes the restrictions on the values of the horizontal blur u_y, which can be any real number in Eq. (3), as well as the value of the vertical component of the blur u_x. Thus, the use of Eq. (3) offers a number of advantages. First, considering the real value of blur within a fraction of a pixel can increase the resolution of details of the restored image when restoring an image at the resolution limit, as shown in [1]. Second, the discreteness of the parameter determining the value of the horizontal blur can be an obstacle when applying the discussed image restoration method as a basis for solving the problem of adaptive deconvolution when the direction and value of the blur are not precisely known.

2. THE 2D DISCRETE PSF AT IMAGE DEFOCUSING

For simplicity, we consider a model where the image is defocused according to the Gaussian law. Although the Gaussian model is not usually used for high quality optical systems such as telescopes and microscopes, it can be used to demonstrate the method for constructing a discrete PSF taking into account the finiteness of pixel sizes. In addition, Gaussian defocus is typically used for demonstration purposes only (e.g., [14]). If required, the Gaussian function can be replaced by any other function, e.g., the Airy function, which corresponds to the case where diffraction is the only cause of defocusing. Here, there are no fundamental restrictions.

Let the function $p(x, y)$ be the intensity of the light flux entering the aperture of the lens. Then, due to the assumed defocus of this flux, the luminance field of points on the light-sensitive panel is defined by the convolution integral

$$q(x,y)=\frac{1}{2\pi(\sigma w)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x-u,y-v)\mathrm{e}^{-\frac{u^2+v^2}{2\pi(\sigma w)^2}}dudv,$$

forming the image during the exposure time τ, where σ is the parameter determining the degree of defocus and w is the pixel size.

Substituting Eq. (1) into this integral, we get the following:

$$q(x,y)=\frac{1}{2\pi\sigma^2w^2}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}p[m,n]\times$$
$$\times\int_{-\infty}^{+\infty}\mathrm{sinc}\left(\frac{x-u}{w}-m\right)\mathrm{e}^{-\frac{u^2}{2\sigma^2w^2}}\times$$
$$\times\,du\int_{-\infty}^{+\infty}\mathrm{sinc}\left(\frac{y-v}{w}-n\right)\mathrm{e}^{-\frac{v^2}{2\sigma^2w^2}}dv$$

Thus, assuming that $x = wl$, $y = wk$, $k,l\in\mathbb{Z}$, we obtain the following:

$$q(k,l)=\frac{1}{2\pi\sigma^2w^2}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}p[m,n]\times$$
$$\times\int_{-\infty}^{+\infty}\mathrm{sinc}\left(k-m-\frac{u}{w}\right)\mathrm{e}^{-\frac{u^2}{2\sigma^2w^2}}du\times$$
$$\times\int_{-\infty}^{+\infty}\mathrm{sinc}\left(1-n-\frac{v}{w}\right)\mathrm{e}^{-\frac{v^2}{2\sigma^2w^2}}dv.$$

Going to the limit at $w \to 0$ in the obtained formula, given that $\mathrm{sinc}(x/w)/w \to \delta(x)$, we arrive at the discrete convolution of the image $p[m, n]$ with the traditional kernel representing a Gaussian grid function. In fact, the discrete convolution kernel traditionally used in this problem is obtained in the limit, as follows:

$$h_{(w \to 0)}[m,n] = \frac{1}{2\pi\sigma^2} \mathrm{e}^{-\frac{m^2+n^2}{2\sigma^2}}.$$

However, without going to the limit, replacing u/w and v/w in the last two integrals by u and v, respectively, gives the following:

$$q(k,l) = \frac{1}{2\pi\sigma^2} \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} p[m,n] \times$$
$$\times \int_{-\infty}^{+\infty} \mathrm{sinc}\left(k-m-u\right) \mathrm{e}^{-\frac{u^2}{2\sigma^2}} du \times$$
$$\times \int_{-\infty}^{+\infty} \mathrm{sinc}\left(1-n-v\right) \mathrm{e}^{-\frac{v^2}{2\sigma^2}} dv.$$

Thus, similar to linear blur, we have a 2D discrete convolution of the form (2), whereas in the case of defocus, the corresponding kernel is separable, as follows:

$$h[m, n] = h_1[m]h_1[n], \tag{6}$$

where

$$h_1[k] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \mathrm{sinc}(k-u) \mathrm{e}^{-\frac{u^2}{2\sigma^2}} du = (\mathrm{sinc} * g)(k),$$

$g(z) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{z^2}{2\sigma^2}}$; here, the asterisk stands for the 1D analogue convolution operation.

We consider the convolution $f(z) = (\mathrm{sinc} * \mathrm{g})(z)$. According to the convolution theorem, the Fourier transform of the function $f(z)$ is the following:

$$F(v) = I_{(-0.5;0.5)}(v) \mathrm{e}^{-2\pi^2 (v\sigma)^2},$$

where it is taken into account that the Fourier image of the function $\mathrm{sinc}(z)$ is an indicator function of the interval $(-0.5; 0.5)$, while the Fourier image of the Gaussian function $g(z)$ is the function $\frac{1}{\pi} \mathrm{e}^{-2(v\sigma)^2}$. To verify the latter, recall that the Fourier image of the function $\mathrm{e}^{-\pi(z/\lambda)^2} / \lambda$ is the function $\mathrm{e}^{-\pi(v/\lambda)^2}$ (in this case, $\lambda = \sqrt{2\pi}\sigma$).

Since $h_1[k]$ is the Fourier original of the function $F(v)$ at the point $z = k$, we have the following:

$$h_1[k] = \int_{-0.5}^{0.5} \mathrm{e}^{-2(\pi v\sigma)^2} \mathrm{e}^{i2\pi vk} dv = \\ = \int_{-0.5}^{0.5} \mathrm{e}^{-2(\pi v\sigma)^2} \cos(2\pi vk) dv. \tag{7}$$

The last equation is due to the fact that the imaginary part of this integral should be zero. This can be verified directly, since there will be an odd function in the imaginary part under the integral. In the limit at $\sigma \to 0$, we have $h_1[k] = \delta[k]$. The graphs of the 1D kernel (7) for different values of the parameter σ are shown in Fig. 2. It can be seen that already at $\sigma = 1.0$ the values of the kernel (7) are practically indistinguishable from the limit values at $w \to 0$.

3. DECONVOLUTION

We consider Eq. (4), which is a finite 2D linear (in each dimension) discrete convolution. The equation is solved using Discrete Fourier Transform (DFT; i.e., 2D DFT). For this, the linear convolution under consideration should first be represented as a cyclic convolution as follows:

$$q[m,n] = \\ = \sum_{k=0}^{M+K-1} \sum_{l=0}^{N+L-1} h[k,l] p[(m-k)_{M+K}, (n-l)_{N+L}], \tag{8}$$

where $(m-k)_{M+K}, (n-l)_{N+L}$ are modulo $M + K$ and modulo $N + L$ residuals, respectively, $m \in \overline{0,(M+K-1)}$, $n \in \overline{0,(N+L-1)}$, and all arrays are assumed to be of equal size $(M + K) \times (N + L)$. This requires adding M null rows and N null columns to the array $h[:, :]$, and K null rows and L null columns to the array $p[:, :]$ (null rows and columns can be added, for example, to the number of the last rows and columns).

Then, according to the discrete cyclic convolution theorem, we get the following:

$$Q[m, n] = H[m, n]P[m, n], \tag{9}$$

where $m \in \overline{0,(M+K-1)}$, $n \in \overline{0,(N+L-1)}$; $H[:, :] = \mathrm{fft}(h[:, :])$, $Q[:, :] = \mathrm{fft}(q[:, :])$, $P[:, :] = \mathrm{fft}(p[:, :])$ are the 2D DPFs of the corresponding arrays.

The problem of reversing the convolution consists in solving Eq. (8) with respect to the array $p[:, :]$, given the array $q[:, :]$. This task is known to be ill-conditioned, i.e., very sensitive to errors in the original data, as well as to noise. Therefore, it is not possible to use the Eq. (9) directly for its solution; rather, it is necessary to use special regularization methods [2–8]. We use the A.N. Tikhonov regularization method, which considers

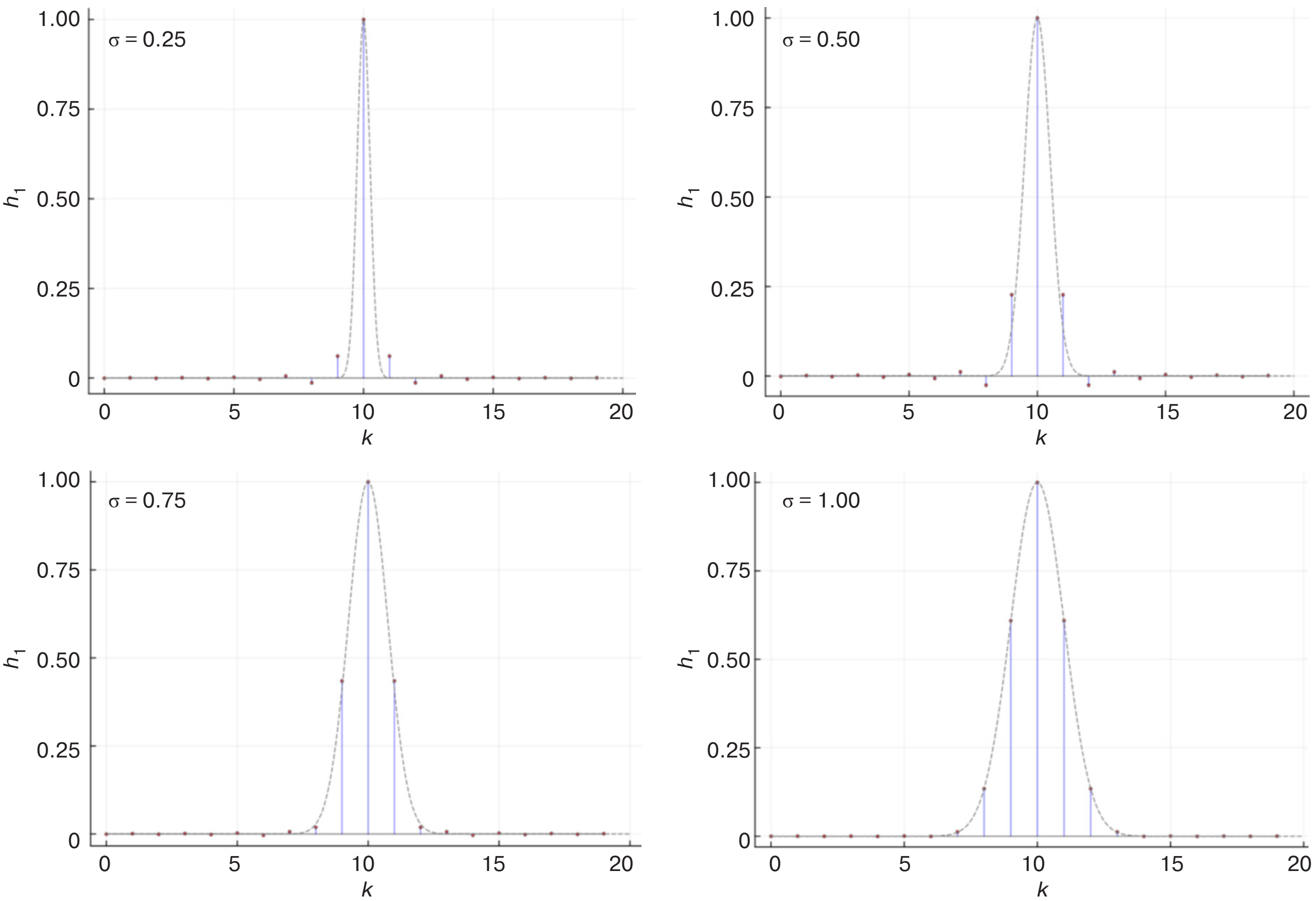


Fig. 2. Examples of 1D kernel graphs (normalized to the maximum) corresponding to the Gaussian defocus; the dashed line shows the plots of the Gaussian curves giving the kernel limits at $w \to 0$

the reciprocal formula with a regularization term instead of directly reversing Eq. (9), as follows:

$$P[m,n] = \frac{\overline{H[m,n]}}{|H[m,n]|^2 + \alpha(R[m,n])^s} Q[m,n], \quad (10)$$

where $\alpha \geq 0$ is the regularization parameter to choose for maximum restored image quality ($\alpha = 0$ means no regularization), $R[:, :]$ is an array corresponding to a chosen regularization function, and $s \geq 0$ is the regularization order. In each case, the regularization function and order are chosen individually.

For example, the regularizing array $R[:, :]$ can be calculated as follows:

$$R[m, n] = R_1[m] + R_2[n], \quad (11)$$

where

$$R_1[m] = \\ = \begin{cases} \pi\left(\dfrac{m}{M+K}\right)^2, & \overline{m \in 0, (M+K)/2-1}, \\ R_{M+K}\left[m - \dfrac{M+K}{2}\right], & \overline{m \in (M+K)/2-1, (M+K)-1}, \end{cases}$$

$$R_2[n] = \\ = \begin{cases} \pi\left(\dfrac{n}{N+L}\right)^2, & \overline{n \in 0, (N+L)/2-1}, \\ R_{N+L}\left[n - \dfrac{N+L}{2}\right], & \overline{n \in (N+L)/2-1, (N+L)-1} \end{cases}$$

(if one of the numbers here, $M+K$ or $N+L$, is odd, then dividing that number by 2 means the integer part of such a division), or as follows:

$$R[:, :] = \mathrm{fft}(\Delta[:, :]), \quad (12)$$

where $\Delta[:, :]$ is some difference approximation of the 2D Laplace differential operator (expanded to a matrix of the desired size with zero rows and columns). The regularization order s is usually chosen low: $s = 0, 1, 2$.

It should be noted that, since we are always restoring an image of finite size, the so-called “edge effect” is inevitable. This is due to the fact that the real image to be restored does not have edges with smoothly decreasing brightness, which are always obtained when modeling a blurred or defocused image (when blurring an image of finite size). Therefore, when modeling such an image, the smoothly decreasing edges should first be cut off in order to make the image resemble reality. Additionally, prior to reconstruction, its edges should be restored or smoothed in some way. Otherwise, the reconstructed image may contain strong artefacts in the form of the so-called Gibbs effect.

4. MODELING RESULTS

The original image used to model the defocused image, the resulting defocused image and the result of restoring it using the kernel that considers finite pixel sizes are shown in Fig. 3. The image with the larger pixel size is shown in Fig. 4. The results of deconvolution using two different PSFs are shown in Fig. 5, where the first does not consider pixel size finiteness (Fig. 5a), while the second does (Fig. 5b). The Gaussian defocus parameter is chosen such that there are noticeable differences between the graphs of the two PSFs. Comparing the results shown in Fig. 5, it is clear that the PSF taking into account the finiteness of the pixel size produces a significantly sharper image.

A similar result is shown in Fig. 6, which shows the reconstruction of the image linearly blurred in a given direction (6 pixels horizontally and 2 pixels vertically) using a kernel that accounts for the finiteness of the pixel size. A series of reconstructed images of different images with different errors in the parameters of the reconstruction kernel that determine the estimated blur vector is shown in Fig. 7. Here, the error values are 25%, 12.5%, 6%, and 0% of the true blur components. It can be seen that, firstly, there may exist situations where error values, even when expressed in fractions of pixels, can significantly worsen the result of the image restoration. Secondly, a successive monotonic reduction of the error values provides a monotonic improvement in image quality. This suggests the possibility of optimizing the parameters of the kernel used to solve the adaptive deconvolution problem.

(a)

(b)

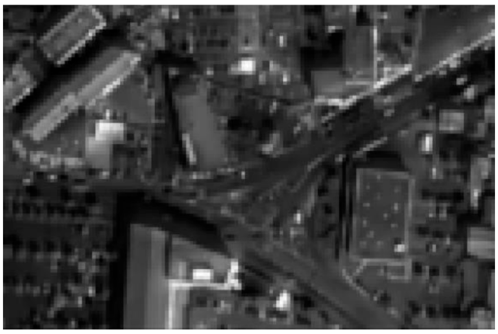
(c)

Fig. 3. Reference image and its Gaussian defocus at σ = 2 along with the result of the convolution with the regularization parameter $\alpha = 10^{-5}$ and the regularization order $s = 1$

(a)

(b)

Fig. 4. Reference image (double grain size compared to Fig. 3) and its Gaussian defocus at σ = 0.4

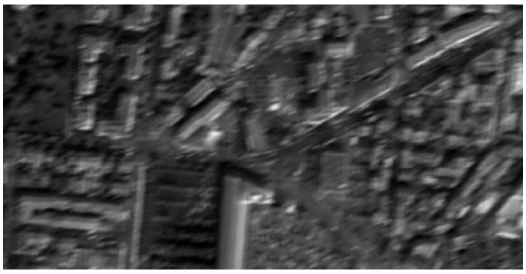


(a)

(b)

Fig. 5. Convolution results of the defocused image from Fig. 4 with regularization parameter α = 10^{-5} and regularization order *s* = 1

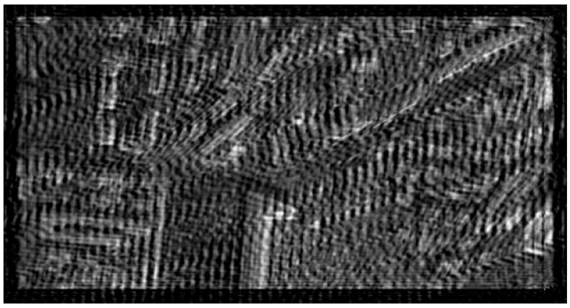
(a)



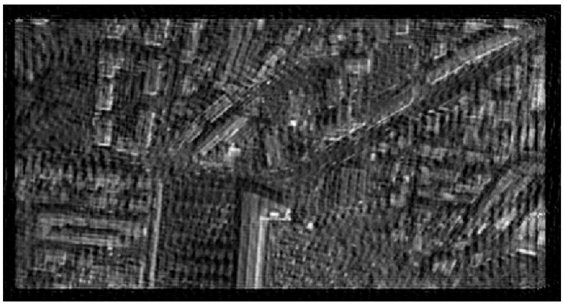
(b)

(c)

Fig. 6. Reference image, its linear blur, and the result of its restoration with regularization parameter α = 10^{-3} and regularization order *s* = 1



(a)



(b)

(c)

(d)

Fig. 7. Reconstruction results of the linearly blurred image from Fig. 6 with successively decreasing error of blur parameters, regularization parameter α = 10^{-3}, and regularization order *s* = 1

CONCLUSIONS

The numerical modeling demonstrates the good performance of the proposed method offering the following advantages. Firstly, by taking into account the finiteness of the pixel size—or more precisely, taking into account the blur parameters within a fraction of a pixel—the resolution of the image details can be improved when the pixel size limit is reached. Importantly, this is achieved without image interpolation. Secondly, the resulting convolution kernel equation for the linear blur makes it possible to recover the image blurred at any angle, not only horizontally. This does not require any prior image rotation to reduce the problem to restoring the horizontally blurred image. In this case, the blur values can be quite high, for example several tens of pixels. Thirdly, since the equation permits the use of blur parameters that are not necessarily expressed in terms of the integer number of pixels, a convenient opportunity arises to use it in solving the adaptive deconvolution problem, where its continuous dependence on both blur parameters may be required.

Authors' contributions

V.B. Fedorov—idea and theoretical part of the study.

S.G. Kharlamov—development of algorithms and conducting computer calculations.

A.V. Fedorov—processing results and assistance in computer calculations.

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