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RESEARCH ARTICLE

Method for estimating objective function landscape convexity during extremum search

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Abstract

Objectives. The work set out to develop a method for estimating the objective function (OF) landscape convexity in the extremum neighborhood. The proposed method, which requires no additional OF calculations or complicated mathematical processing, relies on the data accumulated during extremum search.

Methods. Landscape convexity is characterized by the index of power approximation of the OF in the vicinity of the extremum. The estimation of this index is carried out for pairs of test points taking into account their distances to the found extremum and OF values in them. Based on the analysis of estimation errors, the method includes the selection of test points by their distances from the found extremum and the selection of pairs of test points by the angle between the directions to them from the found extremum. Test functions having different convexities, including concave, were used to experimentally validate the method. The particle swarm optimization algorithm was used as an extremum search method. The experimental results were presented in the form of statistical characteristics and histograms of distributions of the estimation values of the degree of the OF approximation index.

Results. The conductive experiments confirm that the proposed method provides a reliable estimation of power index range bounds upon condition of appropriate definition of trial points and trial point pair selection parameters.

Conclusions. The proposed method may be a part of OF landscape analysis. It is necessary to complement it with the algorithms for automatic adjustment of trial points and pairs of trial points selection parameters. Additional information may be provided by analyzing the dependencies of power index estimations and trial point distances from extrema.

Keywords: objective function landscape, convex function, concave function, power approximation, power index, histogram

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НАУЧНАЯ СТАТЬЯ

Метод оценки выпуклости рельефа целевых функций в процессе поиска экстремума

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Резюме
Цели. Целью работы является разработка метода оценки выпуклости рельефа целевой функции (ЦФ) в окрестностях экстремума, не требующего выполнения дополнительных расчетов ЦФ и сложной математической обработки, а использующего только данные, собираемые в процессе поиска экстремума.
Методы. Выпуклость рельефа характеризуется показателем степени степенной аппроксимации ЦФ в окрестностях экстремума. Оценка этого показателя осуществляется по парам пробных точек с учетом их расстояний до найденного экстремума и значений ЦФ в них. На основе анализа погрешностей такой оценки в методе предусмотрены отбор пробных точек по их расстояниям от найденного экстремума и отбор пар пробных точек по углу между направлениями на них из найденного экстремума. Для экспериментальной проверки метода использовались тестовые функции с различной выпуклостью, как выпуклые, так и вогнутые. В качестве метода поиска экстремума применялся алгоритм роя частиц (particle swarm optimization, PSO). Результаты экспериментов представлялись в виде статистических характеристик и гистограмм распределений значений оценки показателя степени степенной аппроксимации ЦФ.
Результаты. Эксперименты показали, что при соответствующем выборе параметров отбора пробных точек и их пар метод дает достоверные значения границ диапазона, в который попадают оценки показателя степени степенной аппроксимации.
Выводы. Предложенный метод может стать частью методики анализа свойств рельефа ЦФ. Для этого необходимо дополнить его алгоритмами автоматической настройки параметров отбора пробных точек и их пар. Повышение информативности метода может быть достигнуто путем анализа распределения оценок показателя степени по расстояниям пробных точек от экстремума и направлениям на них.

Ключевые слова: рельеф целевой функции, выпуклая функция, вогнутая функция, степенная аппроксимация, показатель степени, гистограмма

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INTRODUCTION

One of the most promising directions for the development and improvement of methods for searching for optimal solutions involves the study of the landscape properties of the optimized target objective functions (OFs) and a consideration of these properties when selecting a search algorithm or/and tuning its parameters [1]. This direction is usually referred to as exploratory landscape analysis (ELA). ELA methods are based on a definition and classification of the OF landscape properties and the development of algorithms for their quantitative evaluation by processing the results of OF calculations at trial points [2–5].

In this paper, we will be interested in the convexity characteristics of landscape properties, according to which the OF landscape areas can be divided into convex and concave ones.

Let us give the definitions [6, 7]. Function $f(\mathbf{x})$ is called convex on the set X if for $\forall(\mathbf{x}_1, \mathbf{x}_2) \in \mathrm{X}$ and $\forall\lambda \in [0,1]$ the following condition is satisfied:

$$f(\mathbf{x}_\lambda) \le \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2), \tag{1}$$

where $\mathbf{x}_\lambda = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$.

Function $f(\mathbf{x})$ is called strictly convex if the inequality in condition (1) is satisfied strictly. Function $f(\mathbf{x})$ is called concave if the function $-f(\mathbf{x})$ is convex. A strictly concave function is defined similarly. The characteristics of convexity are important for understanding the properties of OFs. In particular, if the function is concave in the neighborhood of the minimum point, such a minimum will be unstable in the sense that an insignificant shift from this point can lead to a significant increase in the value of the OF [6, 8].

The set of ELA properties includes convexity characteristics. The methodology of their estimation is as follows [2, 3]. In the search area, a set of trial points $\{\mathbf{x}_i\}$ is formed, where the values of OF $f(\mathbf{x}_i)$ are determined. From this set, pairs of points $\{\mathbf{x}_{j1}, \mathbf{x}_{j2}\}$ are randomly selected, for which the value of $f(\mathbf{x}_{j\lambda})$ at $\lambda = 0.5$ is determined, after which the difference Δ of the left and right parts of (1) is calculated. Next, the convexity probability of the OF is defined as the fraction of pairs of points for which $\Delta < \Delta_{\mathrm{conv}}$, where $\Delta_{\mathrm{conv}} < 0$ is a given threshold. Such a property characterizes the OF on average over the entire search area, rather than individual landscape regions, in particular, the neighborhoods of local extrema, which are of most interest. In addition, to obtain each value of $f(\mathbf{x}_{j\lambda})$ it is required to perform an additional calculation of OF, which in cases where such a calculation is performed by modeling the object, as in many optimization problems of the characteristics of radio engineering devices [9], may require significant time consumption.

In cases where the calculation of the OF gradient is performed, the convexity of the OF can be checked at each iteration by fulfilling the inequality [7]:

$$(\mathbf{x}_2 - \mathbf{x}_1)^{\mathrm{T}} \cdot \left(\nabla f(\mathbf{x}_2) - \nabla f(\mathbf{x}_1)\right) > \varepsilon, \tag{2}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the coordinate vectors of the initial and final iteration points; T is the transpose operation; $\nabla f(\mathbf{x})$ the OF gradient at $\mathbf{x}$; ε is a small positive number.

Calculation of the gradient requires analytical expressions for partial derivatives of the OF on coordinates or application of the finite difference method. In the latter case, the number of OF calculations that require to be performed increases significantly.

The convexity of the OF landscape is also characterized by the eigenvalues of the hessian $\nabla^2 f(\mathbf{x})$—the matrix of second partial derivatives. The function is convex if all eigenvalues of the Hessian are nonnegative. The convexity of the landscape is characterized by absolute values of the eigenvalues along the corresponding directions. In [3], a set of properties determined by the statistics of the ratio of the maximum and minimum eigenvalues of the hessian is introduced. In [10], a measure of the degree of convexity in the form of the number of nonnegative eigenvalues is proposed. However, the computation of the hessian requires a significant number of additional calculations of the OF values.

In recent years, the use of so-called *surrogate* OF models for solving optimization problems has attracted much attention. While such a model should preserve the most important properties of the OF for the extremum search algorithm, the calculation of the values of the modeling function should require significantly less time than determining the value of the OF itself [11, 12]. A sufficiently accurate OF model will also correctly reproduce the convexity of the landscape. Although this approach has excellent prospects, the construction of corresponding models is associated with a large number of calculations.

The task of this work is to develop a method for estimating the convexity of the OF landscape during the search for extrema, which does not require the calculation of the OF derivatives and additional calculations of the OF values beyond those performed by the search algorithm itself, as well as does not require the construction of the surrogate OF models.

ANALYSIS OF THE METHOD FOR ESTIMATING THE CONVEXITY OF THE OF LANDSCAPE

Let us consider the problem of estimating the convexity characteristics of the OF landscape $f(\mathbf{x})$ in the vicinity Ω_{X} of the local minimum $\mathbf{x}^*$, where the following condition is satisfied:

$$f(\mathbf{x}) > f(\mathbf{x}^*), \forall\mathbf{x} \in \Omega_{\mathrm{X}}. \tag{3}$$

We will search for a degree approximation of the OF changes in the vicinity of $\mathbf{x}^*$ in the form:

$$f(\mathbf{x}) - f(\mathbf{x}^*) \approx \hat{f}(\mathbf{x}) = k\left\|\mathbf{x} - \mathbf{x}^*\right\|^{\alpha}, \tag{4}$$

where $\|\mathbf{x}\|$ is the Euclidean norm of the vector $\mathbf{x}$. The index of degree α is an objective characteristic of the convexity of the OF landscape. At $\alpha > 1$, OF is convex, while at $\alpha < 1$, it is concave.

However, the index α does not depend on the value of OF $f(\mathbf{x}^*)$ at the point of extremum, because when this value changes by the same amount, the values of OF at other points will also shift. Therefore, in order to simplify the record, we will assume $f(\mathbf{x}^*) = 0$ without loss of generality and consider (4) as an approximation of the OF itself.

Suppose that the point $\mathbf{x}^*$ is known, the OF is indeed a power function of the form (4), and the values of α and k are the same at all points of Ω_X. Let there be two trial points $\mathbf{x}_1$ and $\mathbf{x}_2$ and the values of the OF at them are $f(\mathbf{x}_1)$, $f(\mathbf{x}_2)$ respectively. Then from the system of equations

$$\begin{cases} f(\mathbf{x}_1) = k\left\|\mathbf{x}_1 - \mathbf{x}^*\right\|^{\alpha}, \\ f(\mathbf{x}_2) = k\left\|\mathbf{x}_2 - \mathbf{x}^*\right\|^{\alpha} \end{cases} \tag{5}$$

we find:

$$\alpha = \frac{\ln\left(f(\mathbf{x}_1)\right) - \ln\left(f(\mathbf{x}_2)\right)}{\ln\left(\left\|\mathbf{x}_1 - \mathbf{x}^*\right\|\right) - \ln\left(\left\|\mathbf{x}_2 - \mathbf{x}^*\right\|\right)}. \tag{6}$$

If the above assumptions are not fulfilled, this estimate will be approximate. Let us estimate the errors arising in this case.

Suppose that the local minimum position $\mathbf{x}'$ found in the search process differs from the true position $\mathbf{x}^*$ (Fig. 1):

$$\mathbf{x}' = \mathbf{x}^* + \Delta\mathbf{x}. \tag{7}$$

In this case we have the estimation:

$$\hat{\alpha} = \frac{\ln\left(f(\mathbf{x}_1)\right) - \ln\left(f(\mathbf{x}_2)\right)}{\ln\left(\left\|\mathbf{x}_1 - \mathbf{x}'\right\|\right) - \ln\left(\left\|\mathbf{x}_2 - \mathbf{x}'\right\|\right)}. \tag{8}$$

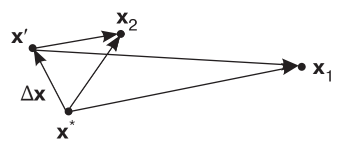

Fig. 1. Analysis of errors at inaccurate determination of the position of the OF minimum

By dividing (8) by (6) and expressing the distances from trial points $\mathbf{x}_1$, $\mathbf{x}_2$ to the true minimum $\mathbf{x}^*$ through the known distances to $\mathbf{x}'$ using the cosine theorem, we get:

$$K_\alpha = \frac{\hat{\alpha}}{\alpha} = \frac{\ln\left(\left\|\mathbf{x}_1 - \mathbf{x}^*\right\|\right) - \ln\left(\left\|\mathbf{x}_2 - \mathbf{x}^*\right\|\right)}{\ln\left(\left\|\mathbf{x}_1 - \mathbf{x}'\right\|\right) - \ln\left(\left\|\mathbf{x}_2 - \mathbf{x}'\right\|\right)} =$$
$$= \frac{0.5\ln\left(\left\|\mathbf{x}_1 - \mathbf{x}'\right\|^2 + \left\|\Delta\mathbf{x}\right\|^2 - 2\left\|\mathbf{x}_1 - \mathbf{x}'\right\|\cdot\left\|\Delta\mathbf{x}\right\|\cdot\cos\psi_1\right)}{\ln\left(\left\|\mathbf{x}_1 - \mathbf{x}'\right\|\right)} - \tag{9}$$
$$- \frac{0.5\ln\left(\left\|\mathbf{x}_2 - \mathbf{x}'\right\|^2 + \left\|\Delta\mathbf{x}\right\|^2 - 2\left\|\mathbf{x}_2 - \mathbf{x}'\right\|\cdot\left\|\Delta\mathbf{x}\right\|\cdot\cos\psi_2\right)}{\ln\left(\left\|\mathbf{x}_2 - \mathbf{x}'\right\|\right)}.$$

Here ψ_1 and ψ_2 are the angles between the vectors $(\mathbf{x}_1 - \mathbf{x}')$, $(\mathbf{x}_2 - \mathbf{x}')$ and the vector $\Delta\mathbf{x}$, respectively.

The value of K_α, which does not depend on the values of OFs in the trial points, is invariant to changes in the scale of distance measurements, making it a convenient characteristic of the estimation $\hat{\alpha}$ error. We will assume that $\left\|\mathbf{x}_1 - \mathbf{x}'\right\| > \left\|\mathbf{x}_2 - \mathbf{x}'\right\|$ and normalize all distances to $\left\|\mathbf{x}_2 - \mathbf{x}'\right\|$. Figure 2 shows the results of calculating by (9) the dependencies of the value of K_α on the distance $\left\|\Delta\mathbf{x}\right\|$ from the true to the found position of the minimum for several combinations of parameters given in Table 1. This assumption is based on the fact that, as will be seen from the following analysis, the angles between the directions to the sample points must be sufficiently small to obtain reliable estimates $\hat{\alpha}$.

Table 1. Parameters of examples of calculation of the K_α dependence on the distance $\|\Delta\mathbf{x}\|$

Examples	$\|\mathbf{x}_1 - \mathbf{x}'\|$	$\|\mathbf{x}_2 - \mathbf{x}'\|$	ψ_1	ψ_2
Example 1	10	1	90	90
Example 2	10	1	100	80
Example 3	10	1	80	100
Example 4	10	1	30	30
Example 5	10	1	150	150
Example 6	3	1	90	90
Example 7	30	1	90	90

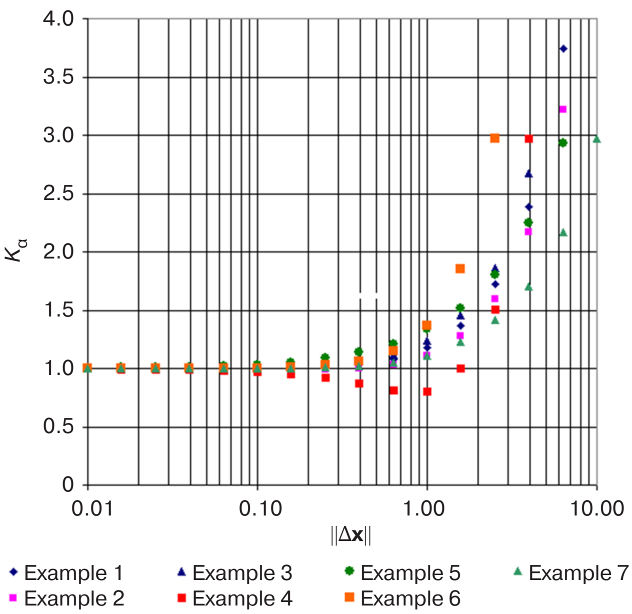

Fig. 2. Dependencies of the ratio K_α of the estimate $\hat{\alpha}$ to the true value of α on the distance $\|\Delta\mathbf{x}\|$ from the true to the found minimum position

The above results allow us to conclude that the error of the index estimation is small in cases when the distances to both trial points are significantly larger than the distance from the true position to the found position of the minimum. More specifically, when the inequality $\|\Delta\mathbf{x}\| \le 0.1\|\mathbf{x}_2 - \mathbf{x}'\|$ is satisfied, the deviation of K_α from unity does not exceed 0.1, which can be considered acceptable for approximate estimation of the convexity of the OF landscape.

Next, we consider the estimation $\hat{\alpha}$ error due to the differences in the values of α_1 and α_2, as well as k_1 and k_2 along the directions from the point of minimum $\mathbf{x}^*$ to the points $\mathbf{x}_1$ and $\mathbf{x}_2$. From (6) we obtain:

$$\begin{aligned}\hat{\alpha} &= \frac{\ln(f(\mathbf{x}_1)) - \ln(f(\mathbf{x}_2))}{\ln(\|\mathbf{x}_1 - \mathbf{x}^*\|) - \ln(\|\mathbf{x}_2 - \mathbf{x}^*\|)} = \\ &= \frac{\ln\left(k_1\|\mathbf{x}_1 - \mathbf{x}^*\|^{\alpha_1}\right) - \ln\left(k_2\|\mathbf{x}_2 - \mathbf{x}^*\|^{\alpha_2}\right)}{\ln(\|\mathbf{x}_1 - \mathbf{x}^*\|) - \ln(\|\mathbf{x}_2 - \mathbf{x}^*\|)} = \\ &= \bar{\alpha} + \frac{\ln(k_1/k_2)}{\ln(\|\mathbf{x}_1 - \mathbf{x}^*\|/\|\mathbf{x}_2 - \mathbf{x}^*\|)} - \\ &- \frac{\Delta\alpha \ln(\|\mathbf{x}_1 - \mathbf{x}^*\| \cdot \|\mathbf{x}_2 - \mathbf{x}^*\|)}{\ln(\|\mathbf{x}_1 - \mathbf{x}^*\|/\|\mathbf{x}_2 - \mathbf{x}^*\|)},\end{aligned} \quad (10)$$

where $\bar{\alpha} = \dfrac{\alpha_1 + \alpha_2}{2}$, $\Delta\alpha = \dfrac{\alpha_2 - \alpha_1}{2}$.

Let us take the arithmetic mean of the indices for the two sample points $\bar{\alpha}$ as the correct estimate of the indicator α. From (10), we obtain the ratio for calculating the absolute error of this estimation.

$$\begin{aligned}E_\alpha = \hat{\alpha} - \bar{\alpha} &= \frac{\ln(k_1/k_2)}{\ln(\|\mathbf{x}_1 - \mathbf{x}^*\|/\|\mathbf{x}_2 - \mathbf{x}^*\|)} - \\ &- \frac{\Delta\alpha \ln(\|\mathbf{x}_1 - \mathbf{x}^*\| \cdot \|\mathbf{x}_2 - \mathbf{x}^*\|)}{\ln(\|\mathbf{x}_1 - \mathbf{x}^*\|/\|\mathbf{x}_2 - \mathbf{x}^*\|)}.\end{aligned} \quad (11)$$

The first summand shows the contribution to the estimation $\hat{\alpha}$ error of the difference in the k coefficients at the two trial points, and the second summand shows the contribution of the difference in the α indices.

Figure 3 shows examples of dependencies of the error magnitude E_α on the distance of the second trial point from the minimum $\|\mathbf{x}_2 - \mathbf{x}^*\|$. The parameters are the distance of the first trial point from the minimum $\|\mathbf{x}_1 - \mathbf{x}^*\|$, as well as the ratio k_1/k_2 and the value Δα introduced above, which characterize the differences of the parameters of the degree approximation at the two points. The values of these parameters for each example are given in Table 2.

Table 2. Parameters of examples of calculation of the E_α dependence on $\|\mathbf{x}_2 - \mathbf{x}^*\|$ values

Examples	$\|\mathbf{x}_2 - \mathbf{x}^*\|$	k_1/k_2	Δα
Example 1	1	2	0
Example 2	1	1	0.2
Example 3	10	1	0.2
Example 4	100	1	0.2
Example 5	10	2	0.2
Example 6	10	0.5	0.2

Example 1 shows the case when the exponent α is constant in all directions, but the coefficient k varies. The error increases with distance $\|\mathbf{x}_2 - \mathbf{x}^*\|$ as the denominator of the first summand decreases. In the next three examples, only the exponent α changes. The dependencies are different for different values of $\|\mathbf{x}_1 - \mathbf{x}^*\|$ due to the fact that the second summand in (11) is not invariant to changes in the scale of distances. The absolute value of E_α with increasing $\|\mathbf{x}_2 - \mathbf{x}^*\|$ can both increase and decrease, or even turn to 0 if the equation $\|\mathbf{x}_1 - \mathbf{x}^*\| \cdot \|\mathbf{x}_2 - \mathbf{x}^*\| = 1$ is satisfied. In the examples presented in rows 5 and 6, both error components are present. The direction of change and the sign of the total error can be different depending on the ratio of parameters.

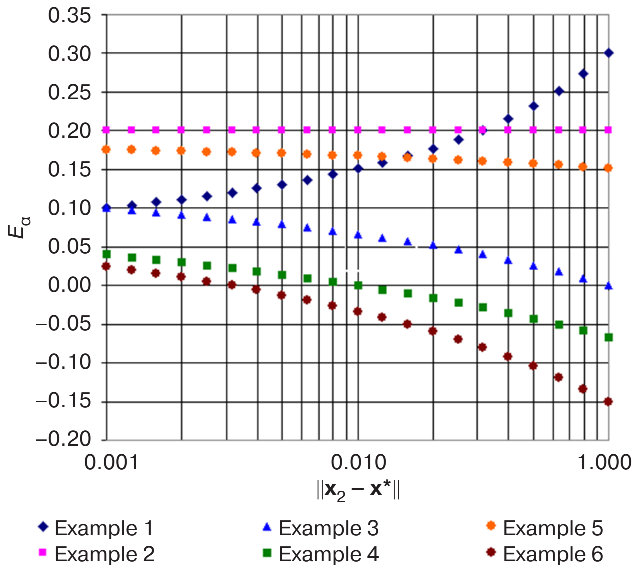

Fig. 3. Dependencies of the difference E_α of the degree index estimation $\hat{\alpha}$ and the accepted as true value $\bar{\alpha}$ on the distance of the nearest sample point to the point of minimum

Thus, the value of the error E_α is affected by the values of the differences between the parameters k and α at the two sample points, and these differences in most cases will be less the smaller the angle between the directions to the sample points from the point of minimum.

The real OF is approximated by a step function of the form (4). In the general case, the approximation will have the form of a step series. Let us consider what information about the convexity of the landscape can be given by the estimation $\hat{\alpha}$ by two trial points. Let the OF be the sum of two degree functions:

$$f(\mathbf{x}) = k_1 \left\|\mathbf{x} - \mathbf{x}^*\right\|^{\alpha_1} + k_2 \left\|\mathbf{x} - \mathbf{x}^*\right\|^{\alpha_2}. \quad (12)$$

The relation (8) takes the form:

$$\hat{\alpha} = \frac{\ln\left(k_1 \left\|\mathbf{x}_1 - \mathbf{x}^*\right\|^{\alpha_1} + k_2 \left\|\mathbf{x}_1 - \mathbf{x}^*\right\|^{\alpha_2}\right)}{\ln\left(\left\|\mathbf{x}_1 - \mathbf{x}^*\right\|\right)} - \frac{\ln\left(k_1 \left\|\mathbf{x}_2 - \mathbf{x}^*\right\|^{\alpha_1} + k_2 \left\|\mathbf{x}_2 - \mathbf{x}^*\right\|^{\alpha_2}\right)}{\ln\left(\left\|\mathbf{x}_2 - \mathbf{x}^*\right\|\right)}. \quad (13)$$

Figure 4 shows examples of dependencies $\hat{\alpha}$ on the distance between the first trial point and the extremum $\|\mathbf{x}_1 - \mathbf{x}^*\|$ for the combinations of parameters given in Table 3.

Table 3. Parameters of examples of calculation of dependence $\hat{\alpha}$ on the values $\|\mathbf{x}_1 - \mathbf{x}^*\|$

Examples	α_1	α_2	k_1	k_2	$\|\mathbf{x}_1 - \mathbf{x}^*\|/\|\mathbf{x}_2 - \mathbf{x}^*\|$
Example 1	1	2	0.5	0.5	10
Example 2	1	2	0.2	0.8	10
Example 3	1	2	0.8	0.2	10
Example 4	1	2	0.5	0.5	3

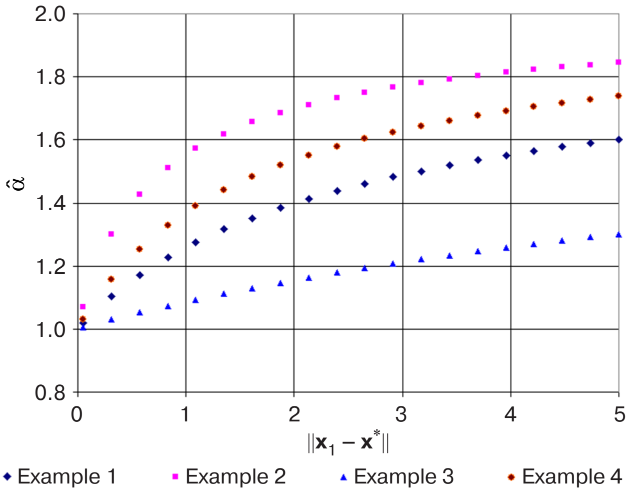

Fig. 4. Dependencies of the degree index $\hat{\alpha}$ estimation on the distance $\|\mathbf{x}_1 - \mathbf{x}^*\|$ at different combinations of parameters in relation (13)

In all the considered examples, the estimation of the degree index $\hat{\alpha}$ changes from a smaller value α_1 to a larger value α_2 as the distances of the reference points to the minimum point increase. The rate of this change depends on the ratios of the weight coefficients k_1, k_2 in (12) (examples 2 and 3), as well as on the ratio of the distances of the two reference points to the minimum point (example 4). Similar regularities will occur with a larger number of summands of the step series. These results should be taken into account when analyzing the convexity of the real OFs.

EXPERIMENTAL

The aim of the experiments was to test the possibility of obtaining reliable estimates $\hat{\alpha}$ using the described method. The methodology of experiments included obtaining sets of trial points in the process of searching for the minimum of the OF and subsequent processing of the collected data to obtain estimates $\hat{\alpha}$ at different parameters of selection of pairs of trial points. The experiments were performed using *MATLAB*[1] programs.

The well-known and widely used particle swarm optimization (PSO) algorithm [13], which, as the experience of its use shows, allows finding extrema of both convex and nonconvex OFs [14], was used as a minimum search method. With the help of this algorithm we searched for the minimum of test functions from the set [15] often used in such studies, as well as specially developed test functions. Information about the test functions will be given below together with the results of experiments. *MATLAB* function implementing the PSO algorithm was modified to return to the program calling it a data array containing the coordinates of all swarm particles in all iterations and the corresponding OF values. Subsequent processing of this data included the following steps:

1. Determination of the coordinates of the found minimum $\mathbf{x}'$ and the value of the OF at this point $f(\mathbf{x}')$.
2. Calculation of distances of all trial points $\mathbf{x}$ from the found minimum $\mathbf{x}'$ and selection by fulfillment of the inequalities $d_{\min} \le \|\mathbf{x} - \mathbf{x}'\| \le d_{\max}$, where $d_{\min}$, $d_{\max}$ are the set thresholds. The value $d_{\min}$ affects the estimation $\hat{\alpha}$ error determined by the relation (9). The value $d_{\max}$ determines the size of the vicinity $\mathbf{x}'$, within which the estimation α is calculated.
3. Calculation of the entropy of the distribution of trial points along the orthants of the coordinate system centered on the point of the found minimum $\mathbf{x}'$. The entropy value is determined by the formula:

[1] https://www.mathworks.com/products/matlab.html. Accessed February 14, 2025.

$$H = -\sum_{i=1}^{Nort} P_i \log_2 P_i, \tag{14}$$

where P_i is the probability of the point getting into the ith orthant; *Nort* is the number of orthants equal to 2^{ND}; *ND* is the dimensionality of the search space. This value gives an estimate of uniformity of distribution of trial points in different directions from the found minimum.

4. Calculation of the angles φ_{ij} between the directions to the trial points $\mathbf{x}_i$, $\mathbf{x}_j$ included in all possible pairs from the previously selected trial points.
5. Selection of pairs of points $\mathbf{x}_i$, $\mathbf{x}_j$ for estimation of the parameters of the degree approximation. The selection conditions are formulated on the basis of the above analysis of errors of the method.

$$\varphi_{ij} \le \varphi_{\max}, \quad \ln \frac{\|\mathbf{x}_i - \mathbf{x}'\|}{\|\mathbf{x}_j - \mathbf{x}'\|} \ge C_1, \tag{15}$$

where $\varphi_{\max}$ and C_1 are the given parameters, and it is assumed that the point $\mathbf{x}_i$ is farther from the found minimum than the point $\mathbf{x}_j$. The value of C_1 determines the minimum of the denominator in (11). The value of $\varphi_{\max}$ determines the maximum angle between the directions to the points of the pair.

6. Calculation of the entropy of the distribution of the selected pairs by orthants, similarly to item 3, which gives an estimate of the completeness of information about the indicator α in different directions.
7. Calculation of estimates of the degree approximation index $\hat{\alpha}$ for the selected pairs of points according to relation (8). Formation of the histogram of the values of these estimates. Calculation of statistical characteristics of their distribution.

Examples of the results of application of the described method are given below. In the cases of isotropic OFs, in which the parameters of the power function (4) are the same in all directions from the minimum, the proposed method finds the values of these parameters with high accuracy. Such examples are not considered here, and attention is paid to anisotropic OFs, for which it is expected that there are errors due to differences in the parameters of the power function in different directions. For all used OFs, the equation $f(\mathbf{x}^*) = 0$ is satisfied, which, as explained earlier, does not lead to a loss of generality of the results.

The data are divided into two tables. Table 4 shows the initial parameters of 12 experiments. The dimensionality of the search space in all experiments is 4. The column "N_{point}" gives the total number of trial points collected during the search for the minimum. The next column gives the distance between the found minimum $\mathbf{x}'$ and the true minimum position $\mathbf{x}^*$. This

Table 4. Initial parameters of the experiments

Exp.	Function	N_{point}	$\|\|\mathbf{x}' - \mathbf{x}^*\|\|$	$d_{\min}$	$d_{\max}$	$\varphi_{\max}$	C_1
1	ellips	1980	$7.11 \cdot 10^{-5}$	$1.00 \cdot 10^{-8}$	10	10	2
2	ellips	1980	$7.11 \cdot 10^{-5}$	0.001	10	10	2
3	ellips	1980	$7.11 \cdot 10^{-5}$	0.001	10	2	2
4	ellips	1980	$7.11 \cdot 10^{-5}$	0.001	10	10	6
5	ellips	1980	$7.11 \cdot 10^{-5}$	0.001	10	2	6
6	diffpowers	1120	$1.03 \cdot 10^{-2}$	$1.00 \cdot 10^{-8}$	10	10	2
7	diffpowers	1120	$1.03 \cdot 10^{-2}$	0.001	10	10	2
8	diffpowers	1120	$1.03 \cdot 10^{-2}$	0.1	10	10	2
9	diffpowers	1120	$1.03 \cdot 10^{-2}$	0.1	10	30	2
10	TestLE4	1420	$1.20 \cdot 10^{-3}$	$1.00 \cdot 10^{-8}$	10	10	2
11	TestLE4	1420	$1.20 \cdot 10^{-3}$	0.01	10	10	2
12	TestLE4	1420	$1.20 \cdot 10^{-3}$	0.1	10	10	2

value is given for reference and is not used by the algorithm since the true position of the minimum is assumed to be unknown. The following columns contain the values of the parameters by which the sample points and their pairs are selected.

Table 5 shows the results of these experiments. Here $N_{\text{sel. point}}$ and $H_{\text{sel. point}}$ are the number of points selected according to item 2 and the entropy of their distribution over orthants, N_{pair}, H_{pair} are the same parameters for pairs of points selected according to item 5. The following columns contain the parameters of the distribution of the estimations $\hat{\alpha}$ for the selected pairs: minimum (min), mean (mean), median (med), maximum (max), standard deviation (std), skewness (skew), and kurtosis (kurt). Histograms of the estimation $\hat{\alpha}$ values for the experiments 5, 9, and 12 are shown in Fig. 5.

Let us proceed to analyze the results of the experiments.

In experiments 1–5, we studied the function ellips($\mathbf{x}$) [15], formed according to the equation:

$$f(\mathbf{x}) = \sum_{n=1}^{ND} \left(x_n - x_n^*\right)^2 \cdot 10^{\left(6(n-1)/(ND-1)\right)}, \quad (16)$$

where $\mathbf{x} = (x_1, ..., x_{ND})$ are the coordinates of the point, $\mathbf{x}^* = (x_1^*, ..., x_{ND}^*)$ are the coordinates of the minimum.

Table 5. Results of experiments

Exp.	$N_{\text{sel.point}}$	$H_{\text{sel.point}}$	N_{pair}	H_{pair}	min	mean	med	max	std	skew	kurt
1	1978	3.845	202413	3.659	0.0003	1.916	1.926	6.545	0.414	0.590	9.351
2	1698	3.775	148938	3.515	0.0003	1.943	1.951	6.545	0.432	0.702	9.199
3	1698	3.775	41786	3.499	0.052	1.951	1.972	4.517	0.301	−0.045	9.988
4	1698	3.775	33095	3.263	0.929	1.940	1.947	3.523	0.231	0.658	7.901
5	1698	3.775	8982	3.222	1.025	1.953	1.969	2.799	0.142	−0.038	7.830
6	1118	3.766	506	3.367	2.447	4.803	4.769	6.564	0.937	−0.381	2.319
7	1118	3.766	506	3.367	2.447	4.803	4.769	6.564	0.937	−0.381	2.319
8	744	3.668	123	3.305	2.755	4.924	4.941	6.249	0.877	−0.382	2.277
9	744	3.668	3373	3.329	2.015	4.562	4.578	6.287	0.954	−0.222	2.271
10	1419	3.706	2448	3.012	0.568	2.615	2.763	3.561	0.435	−2.237	7.655
11	1196	3.710	1078	3.155	0.568	2.534	2.823	3.033	0.578	−1.482	3.981
12	805	3.654	165	2.707	0.568	1.685	1.494	3.016	0.635	0.467	2.355

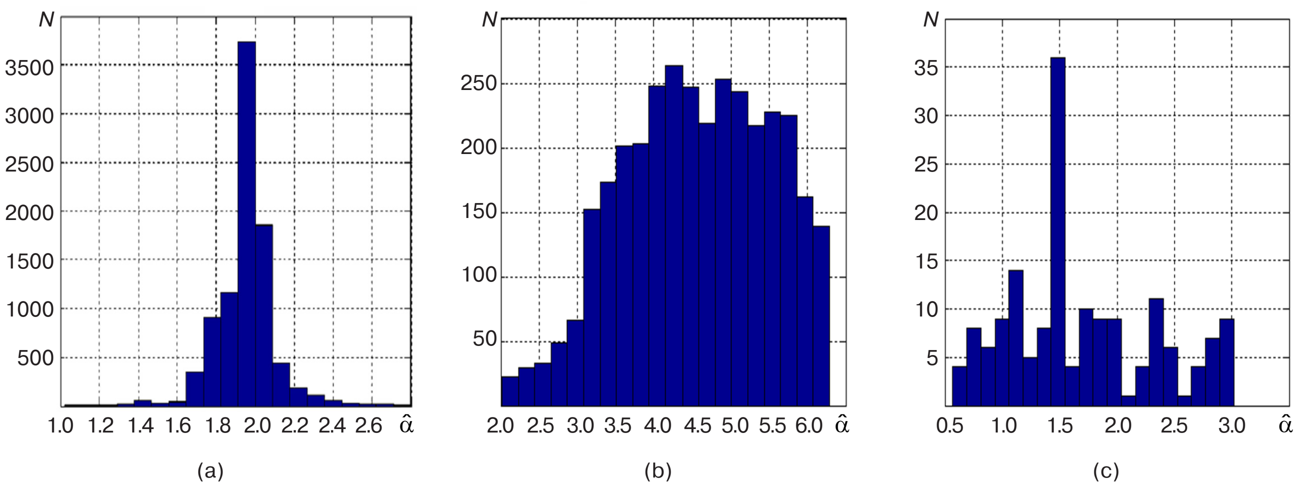

Fig. 5. Histograms of evaluation $\hat{\alpha}$: values: (a) experiment 5, (b) experiment 9, (c) experiment 12

For this OF, the degree exponent $\alpha = 2$ in all directions, and the coefficient k varies in different directions in the range from 1 to 10^6.

In all experiments with this function, the mean and median values of the estimate $\hat{\alpha}$ are close to the correct value of 2. The range of estimates from minimum to maximum narrows as the constraints on pair selection become stronger, and the standard deviation decreases and reaches in experiment 5 a value of about 7% of the mean value, which can be recognized as quite satisfactory. At the same time, the shape of the distribution function of estimates turns out to be symmetric and with a sharp peak (Fig. 5a). The entropy of the distribution of selected points by orthants is close to the maximum value of 4. The entropy of the distribution of the selected pairs is smaller, but from the histogram of this distribution (not given here) we can see that in experiments 1–5 all orthants are represented, i.e., all directions are taken into account in the first approximation. This is also true for the other functions considered below.

In experiments 6–9, the function diffpowers($\mathbf{x}$) [15] defined by the relation:

$$f\left(\mathbf{x}\right)=\sum_{n=1}^{ND}\left(x_n-x_n^*\right)^{\left(2+4\left(n-1\right)/\left(ND-1\right)\right)}, \qquad (17)$$

where the notation is the same as in (16). This function is the sum of degree functions from different components of the point coordinate vector. Degree exponents vary in the range from 2 to 6.

In experiments 6–8, the parameter $d_{\min}$ increases successively, and the number of selected pairs of points decreases. In experiment 7, this leads to narrowing of the range of estimates $\hat{\alpha}$, but in experiment 8, the number of sampled pairs of points becomes too small, and the lower limit of the range is shifted downward. In experiment 9, the tolerance $\varphi_{\max}$ on the angle between the points of a pair is increased. As a result, the number of selected pairs has increased significantly, and the boundaries of the range of estimates $\hat{\alpha}$ (from 2 to 6) are defined with acceptable errors. At the same time, the histogram of $\hat{\alpha}$ values for this experiment is significantly different from zero in the whole range from 2 to 6 (Fig. 5b), which is an indication of the difference of the index in the degree approximation in different directions.

In the standard set of test functions [15] there is no function whose landscape in the region of minimum can be made both convex and concave. To obtain such properties, several additional test functions were developed. Below we present the results of experiments with one of them—TestLE4($\mathbf{x}$) calculated by the following relations:

$$f(\mathbf{x})=k\left\|\mathbf{z}\right\|^{\alpha},$$
$$\mathbf{z}=\mathbf{x}-\mathbf{x}^*,$$
$$k=\frac{1}{\left\|\mathbf{z}\right\|^2}\sum_{n=1}^{ND}\left(K_{1n}z_n^2h(z_n)+K_{2n}z_n^2h(-z_n)\right), \qquad (18)$$
$$\alpha=\frac{1}{\left\|\mathbf{z}\right\|^2}\sum_{n=1}^{ND}\left(W_{1n}z_n^2h(z_n)+W_{2n}z_n^2h(-z_n)\right),$$
$$h(y)=\begin{cases}1, & y>0,\\ 0, & y\le 0.\end{cases}$$

The variables K_{ij} and W_{ij} are elements of matrices $\mathbf{K}$ and $\mathbf{W}$, which have dimensions $2 \times ND$, and represent the values of coefficients and degree exponents, respectively, along the positive and negative directions of all coordinates of the search space. The resulting values of the degree exponent k and coefficient α along the direction to the trial point are obtained by interpolation between the values of these quantities along the coordinate axes. Thus, the possibility of arbitrary setting of the parameters of the degree function along different coordinates and smooth changes of these parameters along intermediate directions is provided.

In experiments 10–12, the following parameter matrices were specified:

$$\mathbf{W}=\begin{pmatrix}3 & 1.5 & 0.5 & 1\\ 1.5 & 2 & 1 & 0.7\end{pmatrix}, \qquad \mathbf{K}=\begin{pmatrix}1 & 2 & 3 & 5\\ 3 & 1 & 0.5 & 1\end{pmatrix}.$$

The function is convex in some directions and concave in others, and the rate of change of the function is also different in different directions. The range of values of the degree exponent is from 0.5 to 3.

In experiments 10–12, the point selection threshold $d_{\min}$ was consistently increased. As a result, the number of selected points and pairs decreased. At the same time, the maximum value of the estimate $\hat{\alpha}$ decreased insignificantly, the minimum value remained unchanged, and the value of the distribution excess decreased significantly, i.e., the distribution became more uniform. The accuracy of estimation of the range $\hat{\alpha}$ boundaries can be considered acceptable. The histogram of estimation values is different from zero in the whole range from the lower to the upper boundaries.

These examples represent a part of the experimental data obtained using different test functions. In addition, besides the PSO algorithm, the differential evolution algorithm [13] and covariance matrix adaptation evolution strategy [16] were used.

CONCLUSIONS

The experimental results confirm the feasibility of the described method to obtain objective information about the convexity of the OF in the neighborhood of the found minimum at appropriate setting of parameters of sampling points and their pairs.

The development of a more detailed method for setting the selection parameters will require further work. One of the possible options in this respect is to automate the process of sequential change of these parameters, rather than performing this operation manually as was done when obtaining the results described above. In this connection, the criteria for selecting parameters can be obtained from statistical characteristics and the shape of the histogram of the distribution of estimates $\hat{\alpha}$. To obtain more information about the convexity of the landscape, in addition to that presented in the above histogram, it is necessary to analyze the distribution of values $\hat{\alpha}$ by distances from the point of the found minimum, as well as the multivariate distribution by distances and directions.

The described method of convexity estimation can become an integral part of the technique of analyzing the OF landscape properties.

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