Mathematical modeling

Математическое моделирование

UDC 539.3; 536.2
https://doi.org/10.32362/2500-316X-2024-12-6-80-90
EDN VWASPO

RESEARCH ARTICLE

Development of model representations of thermal reaction viscoelastic bodies on the temperature field

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Abstract
Objectives. In recent decades, the relevance of research into the thermal response of solids to a temperature field has increased in connection with the creation of powerful energy emitters and their use in technological operations. There is a significant number of publications describing these processes using mathematical models of dynamic or quasi-static thermoelasticity, mainly for most technically important materials that obey Hooke's law. However, at elevated temperatures and higher stress levels, the concept of an elastic body becomes insufficient: almost all materials exhibit more or less clearly the phenomenon of viscous flow. The real body begins to exhibit elastic and viscous properties and becomes viscoelastic. A rather complex problem arises: the development of dynamic (quasi-static) thermoviscoelasticity within the framework of the corresponding mathematical models of classical applied thermomechanics and mathematics. The purpose of the work is to consider the open problem of the theory of thermal shock in terms of a generalized model of thermoviscoelasticity under the conditions of classical Fourier phenomenology on the propagation of heat in solids. Three types of intense heating are considered: temperature, thermal, and medium heating. Intensive cooling modes can be equally considered. The task is posed: to develop model representations of dynamic (quasi-static) thermoviscoelasticity that allow accurate analytical solutions of the corresponding boundary value problems on their basis. This direction is practically absent in the scientific literature.
Methods. Methods and theorems of operational calculus were used.
Results. Model representations of the thermal response of viscoelastic bodies using the proposed new compatibility equation in displacements have been developed.
Conclusions. New integro-differential relations are proposed based on linear rheological models for the Maxwell medium and the Kelvin medium, including both dynamic and quasi-static models for viscoelastic and elastic media, generalizing the results of previous studies. The proposed constitutive relations of the new form are applicable to describe the thermal response of quasi-elastic bodies of a canonical shape simultaneously in three coordinate systems with a system-defining parameter, which makes it possible to identify the influence of the topology of the region on the value of the corresponding temperature stresses.

Keywords: heat stroke, thermoviscoelasticity, generalized dynamic models, analytical solutions, thermal stresses

• **Submitted:** 26.03.2024 • **Revised:** 10.04.2024 • **Accepted:** 10.10.2024

For citation: Kartashov E.M. Development of model representations of thermal reaction viscoelastic bodies on the temperature field. *Russ. Technol. J.* 2024;12(6):80–90. https://doi.org/10.32362/2500-316X-2024-12-6-80-90

Financial disclosure: The author has no financial or proprietary interest in any material or method mentioned.

The author declares no conflicts of interest.

НАУЧНАЯ СТАТЬЯ

Развитие модельных представлений термической реакции вязкоупругих тел на температурное поле

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Резюме
Цели. В последние десятилетия в связи с созданием мощных излучателей энергии и их использованием в технологических операциях возросла актуальность исследований термической реакции твердых тел на температурное поле. Существует значительное количество публикаций, описывающих эти процессы математическими моделями динамической или квазистатической термоупругости, в основном для большинства технически важных материалов, подчиняющихся закону Гука. Однако при повышенных температурах и более высоком уровне напряжений понятие об упругом теле становится недостаточным: почти у всех материалов обнаруживается более или менее отчетливо явление вязкого течения. Реальное тело начинает проявлять упругие и вязкие свойства и становится вязкоупругим. Возникает достаточно сложная проблема – развитие динамической (квазистатической) термовязкоупругости в рамках соответствующих математических моделей классической прикладной термомеханики и математики. Цель работы – рассмотреть открытую проблему теории теплового удара в терминах обобщенной модели термовязкоупругости в условиях классической феноменологии Фурье о распространении теплоты в твердых телах. Рассматриваются три вида интенсивного нагрева: температурный, тепловой, нагрев средой. В равной мере могут быть рассмотрены режимы интенсивного охлаждения. Ставится задача: разработать модельные представления динамической (квазистатической) термовязкоупругости, допускающие точные аналитические решения соответствующих краевых задач на их основе. Указанное направление в научной литературе практически отсутствует.
Методы. Использованы методы и теоремы операционного исчисления.
Результаты. Развиты модельные представления термической реакции вязкоупругих тел с использованием предложенного нового уравнения совместности в перемещениях.
Выводы. Предложены новые интегро-дифференциальные соотношения на базе линейных реологических моделей для среды Максвелла и среды Кельвина, включающие одновременно динамические и квазистатические модели для вязкоупругих и упругих сред, обобщающие результаты предыдущих исследований. Предложенные определяющие соотношения новой формы применимы для описания термической реакции квазиупругих тел канонической формы одновременно в трех системах координат с определяющим систему параметром, что позволяет выявить влияние топологии области на величину соответствующих температурных напряжений.

Ключевые слова: тепловой удар, термовязкоупругость, обобщенные динамические модели, аналитические решения, термические напряжения

• **Поступила:** 26.03.2024 • **Доработана:** 10.04.2024 • **Принята к опубликованию:** 10.10.2024

Для цитирования: Карташов Э.М. Развитие модельных представлений термической реакции вязкоупругих тел на температурное поле. *Russ. Technol. J.* 2024;12(6):80–90. https://doi.org/10.32362/2500-316X-2024-12-6-80-90

Прозрачность финансовой деятельности: Автор не имеет финансовой заинтересованности в представленных материалах или методах.

Автор заявляет об отсутствии конфликта интересов.

INTRODUCTION

The paper continues previous research [1, 2] into the development of generalized local-equilibrium and local-non-equilibrium heat transfer processes. Here, the open problem of the thermal response of viscoelastic bodies to heating of a massive body bounded internally by a flat surface (elastic half-space in the Cartesian coordinate system), a cylindrical surface (elastic space in the cylindrical coordinate system with an internal cylindrical cavity), or a spherical surface (elastic space in the spherical coordinate system with an internal spherical cavity). The developed approach based on integro-differential relations including simultaneously dynamic and quasi-static models for viscoelastic and elastic media generalizes the results of previous studies. New model representations are based on linear Maxwell and Kelvin rheological models, allowing the impact of viscous flow in an elastic medium on temperature elastic stresses to be distinctly traced. The reported results open a new scientific direction in applied thermomechanics and mathematics comprising a study of the thermal response of viscoelastic bodies to intensive heating (cooling) within the framework of dynamic and quasi-static models. At the first stage, the research is carried out under the conditions of the commonly used local equilibrium heat transfer based on the traditional Fourier phenomenology [3] in terms of the linear gradient relations that relate heat flux density vector $\overline{q}(M,t)$ (t is time) with thermal gradient $T(M,t): \overline{q}(M,t) = -\lambda_{\mathrm{T}} \mathrm{qrad} T(M,t)$, where λ_{T} is the thermal conductivity coefficient. Three cases of intensive heating of boundary S of region $\overline{\Omega} = \{M(x,y,z) \in \overline{D} = D + S,\ t > 0\}$ describing a real solid are considered: thermal heating $T(M,t) = T_{\mathrm{am}}(t)$, $M \in S$, $t > 0$ ($T_{\mathrm{am}}(t) > T_0$; T_0 is the initial temperature at which the region is in unstressed and undeformed state); thermal heating $\partial T(M,t)/\partial n = -(1/\lambda_{\mathrm{T}}) q_0(t), M \in S$, $t > 0$ ($q_0(t)$ is heat flux value, $\overline{n} = (n_1, n_2, n_3)$ is external normal to S and is the vector continuous at S); and heating by medium $\partial T(M,t)/\partial n = -h\left[T(M,t) - T_{\mathrm{am}}\right], M \in S$, $t > 0$ (where h – relative heat exchange coefficient; T_{am} – ambient temperature ($T_{\mathrm{am}}(t) > T_0$). Within the described approach, cases of abrupt cooling can also be considered, as well as the effect of heat internal sources (heat sinks).

DEFINING RELATIONS OF DYNAMIC THERMOELASTICITY

Let $\sigma_{ij}(M,t)$, $\varepsilon_{ij}(M,t)$, $U_i(M,t)$ be tensor components of the stress, strain, and displacement vector, respectively, satisfying basic equations of (uncoupled) thermoelasticity (in index notation) [1–6]:

$$\sigma_{ij,j}(M,t) + F_i(M,t) = \rho^* U_i(M,t), \tag{1}$$

$$\varepsilon_{ij}(M,t) = (1/2)\left[U_{i,j}(M,t) + U_{j,i}(M,t)\right], \tag{2}$$

$$\begin{gathered}\sigma_{ij}(M,t) = 2\mu\varepsilon_{ij}(M,t) + \\ + \left[\lambda\varepsilon_{ii}(M,t) - (3\lambda + 2\mu)\alpha_{\mathrm{T}}(T(M,t) - T_0)\right]\delta_{ij}, \\ M \in D,\ t > 0,\end{gathered} \tag{3}$$

where ρ^* is density; λ, μ are Lamé isothermal coefficients; G is the shear modulus; $\lambda = 2Gv/(1-2v)$; v is the Poisson ratio, with $2G(1+v) = E$, E is the Young's modulus; λ_{T} is the linear thermal expansion coefficient, δ_{ij} is the Kronecker symbol, $F_i(M, t)$ are volumetric force components; $e(M, t) = U_{i,i}(M, t) = \varepsilon_{ii}(M, t)$ is the volumetric strain related to the sum of normal stresses $\sigma(M,t) = \sigma_{nn}(M,t), (n = x, y, z)$ described by the following relation:

$$e(M,t) = \frac{1-2v}{E}\sigma(M,t) + 3\alpha_{\mathrm{T}}\left[T(M,t) - T_0\right]. \tag{4}$$

Boundary conditions $\sum_j \sigma_{ji}(M,t) n_j = f_i(M,t)$, $M \in S, t > 0$ should be added to Eqs. (1)–(4) on the part of the surface where stresses are known and boundary conditions $U_i(M,t) = \varphi_i(M,t), M \in S, t > 0$ on the part of the surface where displacements are given. For a partially bounded region, the condition of boundedness of all functions included in (1)–(4) should be added. The temperature function $T(M, t)$ included in (3) is derived from the solution to the boundary value problem of nonstationary thermal conductivity of the following form:

$$\left.\begin{aligned} &\frac{\partial T}{\partial t} = a\Delta T(M,t) + (1/c\rho^*) f(M,t), M \in D,\ t>0,\\ &T(M,t)\big|_{t=0} = T_0, M \in S,\\ &\gamma_1 \frac{\partial T(M,t)}{\partial n} + \gamma_2 T(M,t) = \gamma_3 \varphi(M,t), M \in S,\ t>0, \end{aligned}\right\} \quad (5)$$

where a is thermal diffusivity; c is heat capacity; γ_1, γ_2, and γ_3 are coefficients under the boundary condition.

Relations (1)–(4) are general relations of dynamic thermoelasticity that relate stress, strain, displacement, and temperature. When passing to specific cases, Eq. (1)–(4) should be transformed into the so-called compatibility equations, either in stresses or in displacements, and the corresponding problem of dynamic thermoelasticity should be written for these equations. For the case considered in the paper, the impact of the boundary surface curvature of solid body on the temperature and corresponding temperature stresses should be taken into account. Here, a more convenient mathematical model is the equation of compatibility in displacements that simultaneously covers cylindrical, spherical, and Cartesian coordinate systems only within the framework of the generalized model involving numerous practical applications.

Substituting right parts of (3) into (1) (without volumetric forces) and then using (2) and (4), following a number of long transforms we arrive at the following three equations:

$$\Delta U_i(M,t) + \frac{1}{(1-2\nu)} \cdot \frac{\partial \bar{e}(M,t)}{\partial i} - (\rho^*/G)\frac{\partial^2 U_i(M,t)}{\partial t^2} = \frac{2(1+\nu)\alpha_T}{(1-2\nu)} \frac{\partial\left[T(M,t)-T_0\right]}{\partial i},\ (i = x, y, z),$$

which can be formally written as the following vector equality:

$$\Delta\overline{U}(M,t) + \frac{1}{(1-2\nu)}\operatorname{grad}\left[\operatorname{div}\overline{U}(M,t)\right] - (\rho^*/G)\frac{\partial^2\overline{U}(M,t)}{\partial t^2} = \frac{2(1+\nu)}{(1-2\nu)}\alpha_T \operatorname{grad}\left[T(M,t)-T_0\right], M \in D,\ t>0. \quad (6)$$

Note that during the reverse transition, the appropriate components in vector entries in the left and right parts of (6) should be equated.

We consider further practical cases of dynamic thermoelasticity based on Eq. (6). In the first case, region $z > R$, $t > 0$ is considered in Cartesian coordinates (x, y, z), bounded by flat surface whose temperature state is described by function $T_i(z, t)$, $(i = 1, 2, 3)$; thus, $U_x = U_y = 0$, $U_z = U_z(z, t)$, and Eq. (6) has the following form:

$$\frac{\partial^2 U_z(z.t)}{\partial z^2} - \frac{1}{v_{ew}^2} \cdot \frac{\partial^2 U_z(z,t)}{\partial t^2} = \frac{1+\nu}{1-\nu}\alpha_T \frac{\partial\left[T_i(z,t)-T_0\right]}{\partial z},\ z>R,\ t>0. \quad (7)$$

Here, $v_{ew} = \sqrt{\dfrac{2G(1-\nu)}{\rho^*(1-2\nu)}} = \sqrt{(\lambda+2\mu)/\rho^*}$ is the velocity of the expansive wave (EW) propagation in an elastic medium that is close to the speed of sound.

The stress component $\sigma_{zz}(z, t)$ that interests us is connected to the displacement by the following relation:

$$\sigma_{zz}(z,t) = \frac{2G(1-\nu)}{(1-2\nu)} \times \left\{\frac{\partial U_z}{\partial z} - \frac{1+\nu}{1-\nu}\alpha_T\left[T_i(z,t)-T_0\right]\right\}. \quad (8)$$

The temperature function satisfies three heating conditions:

$$\left.\begin{aligned} &\frac{\partial T_i}{\partial t} = a\frac{\partial^2 T_i}{\partial z^2},\ z>R,\ t>0,\ (i=1,2,3),\\ &T_i(z,t)\big|_{t=0} = T_0,\ z \ge R,\\ &T_1(z,t)\big|_{z=R} = T_{am},\ t>0,\\ &\frac{\partial T_2}{\partial z}\Big|_{z=R} = -(1/\lambda_T)q_0,\ t>0,\\ &\frac{\partial T_3}{\partial z}\Big|_{z=R} = -h(T_3 - T_{am}),\ t>0,\\ &\left|T_i(z,t)\right| < \infty,\ z \ge R,\ t \ge 0. \end{aligned}\right\} \quad (9)$$

In the second case, region $\rho > R$, $t > 0$ having an internal spherical cavity is considered according to spherical coordinates (ρ, φ, θ) when heated under central symmetry conditions $T_i = T_i(\rho, t)$ so that $U_\varphi = U_\theta = 0$, $U_\rho = U_\rho(\rho, t)$, and (6) is written in the following form:

$$\frac{\partial U_\rho(\rho,t)}{\partial \rho^2} + \frac{2}{\rho}\cdot\frac{\partial U_\rho(\rho,t)}{\partial \rho} - \frac{2}{\rho^2}U_\rho(\rho,t) - \frac{1}{v_{ew}^2}\cdot\frac{\partial^2 U_\rho(\rho,t)}{\partial t^2} = \frac{1+\nu}{1-\nu}\alpha_T\frac{\partial\left[T_i(\rho,t)-T_0\right]}{\partial \rho},\ \rho>R,\ t>0. \quad (10)$$

In this case,

$$\sigma_{\rho\rho}(\rho,t)=\frac{2G(1-v)}{(1-2v)}\times$$
$$\times\left\{\frac{\partial U_\rho(\rho,t)}{\partial\rho}+\frac{2v}{1-v}\cdot\frac{1}{\rho}U_\rho(\rho,t)-\right.\qquad(11)$$
$$\left.-\frac{1+v}{1-v}\alpha_T\left[T_i(\rho,t)-T_0\right]\right\},$$

$$\left.\begin{aligned}
&\frac{\partial T_i(\rho,t)}{\partial t}=a\left(\frac{\partial^2 T_i}{\partial\rho^2}+\frac{2}{\rho}\cdot\frac{\partial T_i}{\partial\rho}\right),\ \rho>R,\ t>0,\\
&T_1(\rho,t)\big|_{\rho=R}=T_{am},\ t>0,\\
&\frac{\partial T_2(\rho,t)}{\partial\rho}\Big|_{\rho=R}=-(1/\lambda_T)q_0,\ t>0,\\
&\frac{\partial T_3(\rho,t)}{\partial\rho}\Big|_{\rho=R}=-h\left[T_3(\rho,t)\big|_{\rho=R}-T_{am}\right],\ t>0,\\
&\left|T_i(\rho,t)\right|<\infty,\ \rho\ge R,\ t\ge 0.
\end{aligned}\right\}\qquad(12)$$

In the third case, region $\rho > R$, $t > 0$ with an internal cylindrical cavity is considered in cylindrical coordinates (r, φ, z) under radial temperature conditions $T_i = T_i(\rho, t)$ so that $U_\varphi = U_z = 0$, $U_r = U_r(r, t)$, and Eq. (6) has the following form:

$$\frac{\partial^2 U_r(r,t)}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial U_r(r,t)}{\partial r}-\frac{1}{r^2}U_r(r,t)-$$
$$-\frac{1}{v_{ew}^2}\cdot\frac{\partial^2 U_r(r,t)}{\partial t^2}=\frac{1+v}{1-v}\alpha_T\frac{\partial\left[T_i(r,t)-T_0\right]}{\partial r},\qquad(13)$$
$$r>R,\ t>0.$$

Here,

$$\sigma_{rr}(r,t)=\frac{2G(1-v)}{(1-2v)}\left\{\frac{\partial U_r(r,t)}{\partial r}+\frac{v}{1-v}\cdot\frac{1}{r}U_r(r,t)-\right.\qquad(14)$$
$$\left.-\frac{1+v}{1-v}\alpha_T\left[T_i(r,t)-T_0\right]\right\},$$

$$\left.\begin{aligned}
&\frac{\partial T_i}{\partial t}=a\left(\frac{\partial^2 T_i}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial T_i}{\partial r}\right),\ r>R,\ t>0,\\
&T_i(r,t)\big|_{t=0}=T_0,\ r\ge R,\\
&T_1(r,t)\big|_{r=R}=T_{am},\ t>0,\\
&\frac{\partial T_2(r,t)}{\partial r}\Big|_{r=R}=-(1/\lambda_T)q_0,\ t>0,\\
&\frac{\partial T_3(r,t)}{\partial r}\Big|_{r=R}=-h\left[T_3(r,t)\big|_{r=R}-T_{am}\right],\ t>0,\\
&\left|T_i(r,t)\right|<\infty,\ r\ge R,\ t\ge 0.
\end{aligned}\right\}\qquad(15)$$

It would be useful to simultaneously cover all three cases in all three coordinate systems within the framework of the generalized model, which could be of practical significance in the theory of thermal shock. For convenient recording of the generalized model, the generalized coordinate μ is introduced: $\mu = z$ in Cartesian coordinates, $\mu = \rho$ in spherical coordinates, and $\mu = r$ in cylindrical coordinates. Here, $U_\mu = U_\mu(\mu, t)$, $\sigma_{\mu\mu} = \sigma_{\mu\mu}(\mu, t)$, $T_i = T_i(\mu, t)$.

Then Eqs. (7)–(15) for elastic body can be written in the generalized form, as follows:

$$\frac{\partial^2 U_\mu}{\partial\mu^2}+\frac{2m+1}{\mu}\left(\frac{\partial U_\mu}{\partial\mu}-\frac{1}{\mu}U_\mu\right)-\frac{1}{v_{ew}^2}\cdot\frac{\partial^2 U_\mu}{\partial t^2}=$$
$$=\frac{1+v}{1-v}\alpha_T\frac{\partial}{\partial\mu}\left[T_i(\mu,t)-T_0\right],\ \mu>R,\ t>0,\qquad(16)$$

$$\sigma_{\mu\mu}(\mu,t)=\frac{2G(1-v)}{(1-2v)}\left\{\frac{\partial U_\mu}{\partial\mu}+\frac{(2m+1)v}{(1-v)}\cdot\frac{1}{\mu}U_\mu-\right.\qquad(17)$$
$$\left.-\frac{1+v}{1-v}\alpha_T\left[T_i(\mu,t)-T_0\right]\right\},\ \mu>R,\ t>0,$$

$$\left.\begin{aligned}
&\frac{\partial T_i(\mu,t)}{\partial t}=a\left(\frac{\partial^2 T_i}{\partial\mu^2}+\frac{2m+1}{\mu}\cdot\frac{\partial T_i}{\partial\mu}\right),\ \mu>R,\ t>0,\\
&T_i(\mu,t)\big|_{t=0}=T_0,\ \mu\ge R,\\
&T_1(\mu,t)\big|_{\mu=R}=T_{am},\ t>0,\\
&\frac{\partial T_2(\mu,t)}{\partial\mu}\Big|_{\mu=R}=-(1/\lambda_T)q_0,\ t>0,\\
&\frac{\partial T_3(\mu,t)}{\partial\mu}\Big|_{\mu=R}=-h\left[T_3(\mu,t)\big|_{\mu=R}-T_{am}\right],\ t>0,\\
&\left|T_i(\mu,t)\right|<\infty,\ \mu\ge R,\ t\ge 0.
\end{aligned}\right\}\qquad(18)$$

Here,

$$\mu=\begin{cases}z,\ z>R,\ m=-1/2 & \text{for Cartesian coordinates,}\\ \rho,\ \rho>R,\ m=1/2 & \text{for spherical coordinates,}\\ r,\ r>R,\ m=0 & \text{for cylindrical coordinats.}\end{cases}\qquad(19)$$

In order to completely formulate the dynamic problem for displacements in elastic region (in the latter case, the boundary of the region is assumed stress free), the initial and boundary conditions should be added:

$$U_\mu(\mu,t)\Big|_{t=0},\ \frac{\partial U_\mu(\mu,t)}{\partial t}\Big|_{t=0}=0,\ \mu\ge R,\qquad(20)$$

$$\left[\frac{\partial U_{\mu}(\mu,t)}{\partial \mu}+\frac{(2m+1)\nu}{(1-\nu)}\cdot\frac{1}{\mu}U_{\mu}(\mu,t)\right]\Bigg|_{\mu=R}= \\ =\frac{1+\nu}{1-\nu}\alpha_{\mathrm{T}}\left[T_i(\mu,t)-T_0\right]_{\mu=R},\ t>0, \tag{21}$$

$$\left|U_{\mu}(\mu,t);\sigma_{\mu\mu}(\mu,t)\right|<\infty,\ \mu\geq R,\ t\geq 0. \tag{22}$$

STRESS–STRAIN RELATIONS IN RHEOLOGICAL MODELS

Numerous studies on the thermal response of solids have been carried out mainly for the majority of technically important materials that obey Hooke's law. At relatively low temperature and stress levels, the behavior of a wide class of materials is believed to be in good agreement with the above-described theory of thermoelasticity.

At higher temperatures and stress levels, the concept of an elastic body becomes insufficient due to almost all materials exhibiting more or less distinct viscous flow phenomena. In this case, the behavior of a real body is called viscoelastic since the body simultaneously exhibits elastic and viscous properties. In order to mathematically describe the inelastic behavior of a body under given heating and stress conditions, the stress–strain Eqs. (3) and (4) should be appropriately generalized.

Rheological models that simultaneously account for elastic deformation and viscous flow processes due to the sufficient simplicity of the adopted stress–strain relations permit a mathematical analysis of the behavior of real bodies under different loading conditions. In this connection, when designing structural elements exposed to high temperatures, accounting for the rheological effects becomes of great importance.

We write all necessary relations for the rheological laws relating stresses $\sigma_{ij}(M, t)$ and strains $\varepsilon_{ij}(M, t)$, $(i, j = x, y, z)$. For this, stress deviator $s_{ij}(M, t)$ along with strain deviator $e_{ij}(M, t)$ are introduced by the following relations:

$$s_{ij}(M, t) = \sigma_{ij}(M, t) - \sigma^*(M, t)\delta_{ij}, \tag{23}$$

$$e_{ij}(M, t) = \varepsilon_{ij}(M, t) - \varepsilon^*(M, t)\delta_{ij}, \tag{24}$$

where σ^* and ε^* are average normal stress and average elongation:

$$\sigma^*(M,t)=\frac{1}{3}\sum_i \sigma_{ii}(M,t),\ \ \varepsilon^*(M,t)=\frac{1}{3}\sum_i \varepsilon_{ii}(M,t). \tag{25}$$

Using these deviators, Eqs. (3) and (4) can be written in the following form:

$$s_{ij}(M,t)=2Ge_{ij}(M,t), \tag{26}$$

$$\varepsilon^*(M,t)=\frac{1-2\nu}{2G(1+\nu)}\sigma^*(M,t)+ \\ +\alpha_{\mathrm{T}}\left[T(M,t)-T_0\right]. \tag{27}$$

These equations describe the behavior of a linear elastic medium. Adding the summand expressing Newton's law of viscosity (series or parallel connection of spring and viscous resistance) to Hooke's law relations, the resulting dependencies would yield the Maxwell medium, as follows:

$$\frac{\partial s_{ij}(M,t)}{\partial t}+\frac{1}{\tau_{\mathrm{rlx}}}s_{ij}(M,t)=2G\frac{\partial e_{ij}(M,t)}{\partial t} \tag{28}$$

and the Kelvin medium, as follows:

$$s_{ij}(M,t)=2G\left[e_{ij}(M,t)+\tau_{\mathrm{rlx}}\frac{\partial e_{ij}(M,t)}{\partial t}\right]. \tag{29}$$

In this case, Eq. (27) remains unchanged. The latter means that under hydrostatic compression or tension, the body behaves as a fully elastic body. The constant $\tau_{\mathrm{rlx}} = \eta/G$ is referred to as the relaxation time in (28) and the lag time in (29), while η is the material viscosity. Certainly, an actual behavior of materials is more complicated than in hypothetical cases (28) and (29); however, when based on applying the simplest models, the Maxwell scheme can be used for metals at high temperatures, as well as for polymers combining elastic deformation and viscous flow, while the Kelvin scheme can be used for materials with internal friction in studying damped oscillations.

Note that at $\tau_{\mathrm{rlx}} = 0$ $(\eta = \infty)$, Eq. (28) yields Hooke's medium, while at $\tau_{\mathrm{rlx}} = 0$ $(\eta = 0)$ in (29), Kelvin's law reduces to Eq. (26).

At thermal shock (instant heating or cooling of the boundary surface), the stresses immediately change by value $\Delta=\left|E\alpha_{\mathrm{T}}\left(T_{\mathrm{am}}-T_0\right)\right|$ [3]. In an elastic medium, these stresses remain unchanged, while in a Maxwell medium, viscous flow begins, as a result of which the stress continuously decreases to asymptotically approach a zero value. In contrast, in the Kelvin medium, the stress jump exceeds the appropriate elastic value toward which this stress then approaches asymptotically.

NEW INTEGRAL RELATIONS FOR DYNAMIC THERMOVISCOELASTICITY

Since stress–strain relations for viscoelastic materials contain variable t (time), the corresponding mathematical models are nonstationary and therefore dynamic. The above relations can be used to describe the thermal response of canonically shaped viscoelastic bodies (an infinite plate; a half-space bounded by a flat surface; cylindrical and spherical bodies, and etc.) under given heating (or cooling) conditions as part of the corresponding boundary value problem of nonstationary thermal conductivity. For this purpose, the differential equation of dynamic thermoviscoelasticity should be obtained at the initial stage. We start considering this issue in Cartesian coordinates for viscoelastic half-space $z \geq l$ (l is left boundary of the region) of temperature $T(z, t)$ whose boundary is stress-free. In this case, $U_x = U_y = 0$, $U_z = U_z(z, t)$, $\varepsilon_{xx} = \varepsilon_{yy} = 0$, $e_{zz} = \varepsilon_{zz}$, stresses $\sigma_{ij} = \sigma_{ij}(z, t)$ for $i = j$, $\sigma_{ij} = 0$ for $i \neq j$, $(i, j = x, y, z)$.

Then we have the following:

$$\left.\begin{aligned} &\frac{\partial s_{zz}(z,t)}{\partial t} + \frac{1}{\tau_{\text{rlx}}} s_{zz}(z,t) = \frac{4G}{3} \cdot \frac{\partial \varepsilon_{zz}(z,t)}{\partial t}, \ t > 0, \\ &s_{zz}(z,t)\big|_{t=0} = 0, \end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned} &\varepsilon_{zz}(z,t) = \frac{\partial U_z(z,t)}{\partial z}, \\ &\frac{\partial \sigma_{zz}(z,t)}{\partial z} = \rho \frac{\partial^2 U_z(z,t)}{\partial t^2}, \ z > l, \ t > 0, \end{aligned}\right\} \tag{31}$$

$$\begin{aligned} \sigma_{zz} = s_{zz} + \sigma^* = s_{zz} + \frac{2G(1+\nu)}{3(1-2\nu)} \varepsilon_{zz} - \\ - \frac{2G(1+\nu)}{(1-2\nu)} \alpha_{\text{T}} (T_i - T_0). \end{aligned} \tag{32}$$

We find the solution to the Cauchy problem (30):

$$s_{zz} = \frac{4G}{3} \varepsilon_{zz} - \frac{4G}{3\tau_{\text{rlx}}} \int_0^t \exp\left[-\frac{(t-\tau)}{\tau_{\text{rlx}}}\right] \varepsilon_{zz}(z,\tau) d\tau. \tag{33}$$

Then we find σ_{zz} from (32) and (33) and substitute it into (31). As a result, the following relation for the Maxwell medium is obtained:

$$\begin{aligned} \frac{\partial^2 U_z}{\partial z^2} - \frac{1}{v_{\text{ew}}^2} \cdot \frac{\partial^2 U_z}{\partial t^2} = \frac{(1+\nu)}{(1-\nu)} \alpha_{\text{T}} \frac{\partial \left[T_i(z,t) - T_0\right]}{\partial z} + \\ + \frac{2(1-2\nu)}{3\tau_{\text{rlx}}(1-\nu)} \int_0^t \exp\left[-\frac{(t-\tau)}{\tau_{\text{rlx}}}\right] \frac{\partial^2 U_z(z,\tau)}{\partial z^2} d\tau. \end{aligned} \tag{34}$$

In this case,

$$\begin{aligned} \sigma_{zz}(z,t) = \frac{2G(1-\nu)}{(1-2\nu)} \cdot \frac{\partial U_z}{\partial z} - \\ - \frac{4G}{3\tau_{\text{rlx}}} \int_0^t \exp\left[-\frac{(t-\tau)}{\tau_{\text{rlx}}}\right] \frac{\partial U_z(z,\tau)}{\partial z} d\tau - \\ - \frac{2G(1+\nu)}{(1-2\nu)} \alpha_{\text{T}} \left[T_i(z,t) - T_0\right]. \end{aligned} \tag{35}$$

Using similar reasoning in the spherical coordinate system (central symmetry $T_i = T_i(\rho, t)$ for the viscoelastic region $\rho > R$, $t > 0$, relations for the Maxwell medium are obtained:

$$\begin{aligned} \frac{\partial^2 U_\rho}{\partial \rho^2} + \frac{2}{\rho} \cdot \frac{\partial U_\rho}{\partial \rho} - \frac{2}{\rho^2} U_\rho - \frac{1}{v_{\text{ew}}^2} \cdot \frac{\partial^2 U_\rho}{\partial t^2} = \\ = \frac{1+\nu}{1-\nu} \alpha_{\text{T}} \frac{\partial \left[T_i(\rho,t) - T_0\right]}{\partial \rho} + \frac{2(1-2\nu)}{3\tau_{\text{rlx}}(1-\nu)} \times \\ \times \int_0^t \exp\left[-\tau \frac{(t-\tau)}{\tau_{\text{rlx}}}\right] \left(\frac{\partial^2 U_\rho}{\partial \rho^2} + \frac{2}{\rho} \cdot \frac{\partial U_\rho}{\partial \rho} - \frac{2}{\rho^2} U_\rho(\rho,\tau)\right) d\tau, \end{aligned} \tag{36}$$

$$\begin{aligned} \sigma_{\rho\rho}(\rho,t) = \frac{2G(1-\nu)}{(1-2\nu)} \times \\ \times \left\{ \frac{\partial U_\rho}{\partial \rho} + \frac{2\nu}{1-\nu} \cdot \frac{1}{\rho} U_\rho - \frac{(1+\nu)}{(1-\nu)} \alpha_{\text{T}} \left[T_i(\rho,t) - T_0\right] - \right. \\ \left. - \frac{2}{3\tau_{\text{rlx}}} \int_0^t \exp\left[-\frac{(t-\tau)}{\tau_{\text{rlx}}}\right] \left(\frac{\partial U_\rho}{\partial \rho} - \frac{1}{\rho} U_\rho(\rho,\tau)\right) d\tau \right\}. \end{aligned} \tag{37}$$

In cylindrical coordinates (radial flux $T_i = T_i(r, t)$) for viscoelastic region $r > R$, $t > 0$, similar reasoning produces the following:

$$\begin{aligned} \frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial U_r}{\partial r} - \frac{1}{r^2} U_r - \\ - \frac{1}{v_{\text{ew}}^2} \cdot \frac{\partial^2 U_r}{\partial t^2} = \frac{(1+\nu)}{(1-\nu)} \alpha_{\text{T}} \frac{\partial \left[T_i(r,t) - T_0\right]}{\partial r} + \frac{2(1-2\nu)}{3\tau_{\text{rlx}}(1-\nu)} \times \\ \times \int_0^t \exp\left[-\frac{(t-\tau)}{\tau_{\text{rlx}}}\right] \left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial U_r}{\partial r} - \frac{1}{r^2} U_r(r,\tau)\right) d\tau, \end{aligned} \tag{38}$$

$$\begin{aligned} \sigma_{rr}(r,t) = \frac{2G(1-\nu)}{(1-2\nu)} \times \\ \times \left\{ \frac{\partial U_r}{\partial r} + \frac{\nu}{1-\nu} \cdot \frac{1}{r} U_r - \frac{(1+\nu)}{(1-\nu)} \alpha_{\text{T}} \left[T_i(r,t) - T_0\right] - \right. \\ \left. - \frac{2}{3\tau_{\text{rlx}}} \int_0^t \exp\left[-\frac{(t-\tau)}{\tau_{\text{rlx}}}\right] \left(\frac{\partial U_r}{\partial r} - \frac{1}{2r} U_r(r,\tau)\right) d\tau \right\}. \end{aligned} \tag{39}$$

Thus, the generalized model of dynamic thermoviscoelasticity can be written in coordinates (μ, t) for all three coordinate systems simultaneously.

For the Maxwell medium:

$$\frac{\partial^2 U_\mu}{\partial\mu^2}+\frac{2m+1}{\mu}\left(\frac{\partial U_\mu}{\partial\mu}-\frac{1}{\mu}U_\mu\right)-\frac{1}{v_{\text{ew}}^2}\cdot\frac{\partial^2 U_\mu}{\partial t^2}=\frac{(1+v)}{(1-v)}\alpha_{\text{T}}\frac{\partial\left[T_i(\mu,t)-T_0\right]}{\partial\mu}+\frac{2(1-2v)}{3\tau_{\text{rlx}}(1-v)}\times\int_0^t\exp\left[-\frac{(t-\tau)}{\tau_{\text{rlx}}}\right]\left[\frac{\partial^2 U_\mu}{\partial\mu^2}+\frac{2m+1}{\mu}\left(\frac{\partial U_\mu}{\partial\mu}-\frac{1}{\mu}U_\mu(\mu,\tau)\right)\right]d\tau, \quad (40)$$

$$\sigma_{\mu\mu}(\mu,t)=\frac{2G(1-v)}{(1-2v)}\times\left\{\frac{\partial U_\mu}{\partial\mu}+\frac{(2m+1)v}{(1-v)}\cdot\frac{1}{\mu}U_\mu-\frac{(1+v)}{(1-v)}\alpha_{\text{T}}\left[T_i(\mu,t)-T_0\right]\right\}-\frac{4G}{3\tau_{\text{rlx}}}\int_0^t\exp\left[-\frac{(t-\tau)}{\tau_{\text{rlx}}}\right]\left[\frac{\partial U_\mu}{\partial\mu}-\frac{2m+1}{2\mu}U_\mu(\mu,\tau)\right]d\tau. \quad (41)$$

The specific coordinate system in Eqs. (40) and (41) is fixed by (19).

For the Kelvin medium:

$$\frac{\partial^2 U_\mu}{\partial\mu^2}+\frac{2m+1}{\mu}\left(\frac{\partial U_\mu}{\partial\mu}-\frac{1}{\mu}U_\mu\right)-\frac{1}{v_{\text{ew}}^2}\cdot\frac{\partial^2 U_\mu}{\partial t^2}=\frac{(1+v)}{(1-v)}\alpha_{\text{T}}\frac{\partial\left[T_i(\mu,t)-T_0\right]}{\partial\mu}-\frac{2\tau_{\text{rlx}}}{3}\cdot\frac{(1-2v)}{1-v}\times\frac{\partial}{\partial t}\left[\frac{\partial^2 U_\mu}{\partial\mu^2}+\frac{2m+1}{\mu}\left(\frac{\partial U_\mu}{\partial\mu}-\frac{1}{\mu}U_\mu\right)\right]. \quad (42)$$

$$\sigma_{\mu\mu}(\mu,t)=\frac{2G(1-v)}{(1-2v)}\times\left\{\frac{\partial U_\mu}{\partial\mu}+\frac{v}{1-v}\cdot\frac{2m+1}{\mu}U_\mu-\frac{1+v}{1-v}\alpha_{\text{T}}\left[T_i(\mu,t)-T_0\right]\right\}+\frac{4G\tau_{\text{rlx}}}{3}\cdot\frac{\partial}{\partial t}\left(\frac{\partial U_\mu}{\partial\mu}-\frac{2m+1}{2\mu}U_\mu\right). \quad (43)$$

As in (41) above, the corresponding coordinate system is defined by conditions (19). Functions $T_i(\mu, t)$, (i = 1, 2, 3) correspond to statements (18). For writing boundary value problems for Eq. (40) and (42), initial conditions (20), boundedness conditions (22), and the boundary condition for the boundary of region $\mu \geq R$, $t \geq 0$ free of stresses (41) and (43) should be added. When conducting numerical experiments for different thermal heating (or cooling) conditions specified in (18), Eqs. (40) and (42) admit Laplace transforms that permit passage to linear boundary value problems for displacements in the image space and, after finding them, writing all (nonzero) components of stress and strain tensors out. Following passage to the originals, it becomes possible to reproduce the complete picture of the dynamic response of viscoelastic body to thermal shock. For such purposes, partial differential equations (34), (36), (38) can also be used; moreover, it then becomes possible (which is more interesting) to go straight to generalized models for Eqs. (40) and (42). In [2], the analytical method for finding exact operational solutions to such generalized equations is developed, which ultimately permits a description of the impact of the region topology (by fixing m in the problem solution) on the magnitude of viscoelastic temperature stresses. In practical terms, the latter is of considerable interest for many fields of science and technology [3–6].

Another new approach based on deviatoric relations, which also provides a dynamic formulation of the thermoviscoelastic problem, can be mentioned here. We consider this approach for Cartesian coordinates. From (32) and (33), the following is obtained:

$$\sigma_{zz}(z,t)=\frac{2G(1-v)}{(1-2v)}\varepsilon_{zz}-\frac{2G(1+v)}{(1-2v)}\times\alpha_{\text{T}}\left[T_i(z,t)-T_0\right]-\frac{4G}{3\tau_{\text{rlx}}}\times\int_0^t\exp\left[-\frac{(t-\tau)}{\tau_{\text{rlx}}}\right]\varepsilon_{zz}(z,\tau)d\tau. \quad (44)$$

We use the operational method to find $\bar{\varepsilon}_{zz}(z,p)$ from (44) and substitute the resulting relation into the operational form of the equation as follows:

$$\frac{\partial^2\sigma_{zz}}{\partial z^2}=\rho^*\frac{\partial^2}{\partial t^2}(\varepsilon_{zz}).$$

After long transforms, the following equation of a new type is obtained:

$$\frac{\partial^2\sigma_{zz}}{\partial z^2}-\frac{1}{v_{\text{ew}}^2}\cdot\frac{\partial^2\sigma_{zz}}{\partial t^2}=\frac{1+v}{1-v}\alpha_{\text{T}}\rho^*\frac{\partial^2\left[T_i(z,t)-T_0\right]}{\partial t^2}+\frac{m_1}{v_{\text{ew}}^2\tau_{\text{rlx}}}\times\frac{\partial^2}{\partial t^2}\int_0^t\exp\left[-(m_2/3\tau_{\text{rlx}})(t-\tau)\right]\sigma_{zz}(z,\tau)d\tau+\frac{m_1m_2}{\tau_{\text{rlx}}(1/\rho^*)}\times\frac{\partial^2}{\partial t^2}\int_0^t\exp\left[-(m_2/3\tau_{\text{rlx}})(t-\tau)\right]\alpha_{\text{T}}\left[T_i(z,\tau)-T_0\right]d\tau, \quad (45)$$

$$z>l,\ t>0.$$

Here, $m_1=\frac{2(1-2v)}{3(1-v)}$, $m_2=\frac{1+v}{1-v}$.

Equation (45), which generalizes the well-known Danilovskaya equation for elastic bodies [7] to viscoelastic bodies, provides further development of the above problem (within the Maxwell medium framework). For the Kelvin medium, we have the following equation:

$$\frac{\partial^2\sigma_{zz}}{\partial z^2}=\frac{1}{m_1\tau_{\text{rlx}}v_{\text{ew}}^2}\times$$
$$\times\frac{\partial^2}{\partial t^2}\int_0^t\exp\left[-\frac{(t-\tau)}{m_1\tau_{\text{rlx}}}\right]\sigma_{zz}(z,\tau)d\tau+\frac{m_2\rho^*}{m_1\tau_{\text{rlx}}}\times \quad (46)$$
$$\times\frac{\partial^2}{\partial t^2}\int_0^t\exp\left[-\frac{(t-\tau)}{m_1\tau_{\text{rlx}}}\right]\alpha_{\text{T}}\left[T_i(z,\tau)-T_0\right]d\tau.$$

For numerical calculations, for example, based on Eq. (45), it is reasonable to pass to dimensionless quantities using the following formulas:

$$\xi=\frac{v_{\text{ew}}(z-l)}{a},\ \tau=\frac{v_{\text{ew}}^2t}{a},$$
$$\beta_1=\frac{2(1-2v)}{3(1-v)\tau_{\text{rlx}}(v_{\text{ew}}^2/a)},\ \beta_2=\frac{(1+v)}{(1-v)3\tau_{\text{rlx}}(v_{\text{ew}}^2/a)},$$
$$S_{\text{T}}=\frac{2G\alpha_{\text{T}}(T_{\text{am}}-T_0)(1+v)}{(1-2v)},\ \sigma_{\xi\xi}(\xi,\tau)=\frac{\sigma_{zz}(z,t)}{S_{\text{T}}},$$
$$W_i(\xi,\tau)=\frac{T_i(z,t)-T_0}{T_{\text{am}}-T_0}.$$

Then Eq. (45) takes the following form:

$$\frac{\partial^2\sigma_{\xi\xi}}{\partial\xi^2}-\frac{\partial^2\sigma_{\xi\xi}}{\partial\tau^2}=\frac{\partial^2W}{\partial\tau^2}+$$
$$+\beta_1\int_0^\tau\exp\left[-\beta_2(\tau-\tau')\right]\left[\sigma_{\xi\xi}(\xi,\tau')+W(\xi,\tau')\right]d\tau'. \quad (47)$$

In this form, the equation seems more convenient for Laplace transforms in the image space since it contains a convolution-type summand (which is convenient for applying the Laplace transform).

We find the operational solution of Eq. (47):

$$\bar\sigma_{\xi\xi}(\xi,p)=\frac{1}{2}p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\bar W(0,p)\times$$
$$\times\int_0^\infty\exp\left[-p(\xi+\xi')\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right]d\xi'-$$
$$-\frac{1}{2}p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\bar W(\xi,p)\times$$
$$\times\int_\xi^\infty\exp\left[-p(\xi'-\xi)\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right]d\xi'- \quad (48)$$
$$-\frac{1}{2}p\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\bar W(\xi,p)\times$$
$$\times\int_0^\xi\exp\left[-p(\xi-\xi')\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right]d\xi'.$$

The given representation, which is characteristic for dynamic problems of thermoviscoelasticity, differs from conventional representations (with originals) in tables [8]. The key issue in finding the original of the complex representation (48) is the preliminary obtaining of its origin

$$\bar\Psi_i(\xi,\xi',p)=\frac{1}{p}\exp\left[-\gamma_i(\xi,\xi')\sqrt{\frac{p+\beta_1+\beta_2}{p+\beta_2}}\right]. \quad (49)$$

Here, the approach developed in [2] for complex representations can be used. For this, the Riemann–Mellin integral is applied with allowance for function (49) having two branching points. We omit long calculations and provide the final result:

$$\Psi_i(\xi,\xi',\tau)=\left\{1-\frac{1}{\pi}\int_0^{\beta_1}\frac{1}{x+\beta_2}\exp\left[-(x+\beta_2)\tau\right]\times\right.$$
$$\left.\times\sin\left[\gamma_i(\xi,\xi')(x+\beta_2)\sqrt{\frac{\beta_1-x}{x}}\right]dx\right\}\times \quad (50)$$
$$\times\eta\left[\tau-\gamma_i(\xi,\xi')\right].$$

Here,

$$\gamma_i(\xi,\xi')=\begin{cases}(\xi+\xi'),\ i=1,\\(\xi'-\xi),\ i=2,\\(\xi-\xi'),\ i=3,\end{cases}$$

$\eta(z)$ is the Heaviside function. Then the origin of representation (48) can be written out:

$$\sigma_{\xi\xi}(\xi,\tau)=-W(\xi,\tau)-$$
$$-\frac{1}{2}\cdot\frac{\partial}{\partial\xi}\int_0^\infty d\xi'\int_0^\tau\frac{\partial W(0,\tau')}{\partial\tau'}\Psi_1(\xi,\xi',\tau-\tau')d\tau'-$$
$$-\frac{1}{2}\cdot\frac{\partial}{\partial\xi}\int_\xi^\infty d\xi'\int_0^\tau\frac{\partial W(\xi,\tau')}{\partial\tau'}\Psi_2(\xi,\xi',\tau-\tau')d\tau'+ \quad (51)$$
$$+\frac{1}{2}\cdot\frac{\partial}{\partial\xi}\int_0^\xi d\xi'\int_0^\tau\frac{\partial W(\xi,\tau')}{\partial\tau'}\Psi_3(\xi,\xi',\tau-\tau')d\tau'.$$

Other coordinate systems can be considered similarly.

Finishing this part of the theory of dynamic thermoviscoelasticity, generalized Eqs. (16) and (17) for the elastic medium should be compared with Eqs. (40), (41) for the Maxwell model and (42), (43) for the Kelvin model for a viscoelastic medium. Here, the influence of viscosity and its contribution to generalized thermomechanics is clearly shown. In fact, the above relations (as well as (45), (46), and (51)) open a promising scientific direction related to investigation of the thermal response of viscoelastic media to heating (or cooling) in terms of dynamic viscoelasticity. For example, (51) can consider numerous cases of heating (cooling) in the framework of model problems (9) with different kinds of heat flow: homogeneous, inhomogeneous, pulsed, pulsating, periodic, aperiodic, etc. Each case of this study represents independent scientific research involving not only thermomechanics, but also computational mathematics, and especially operational calculus in finding the origins of complex representations. Here it should be noted that such solutions to dynamic problems are practically not covered in the literature. Further studies of the above problem consist in developing generalized model representations of the thermal reaction of viscoelastic media for locally nonequilibrium heat transfer processes [9–15].

CONCLUSIONS

In the paper, new model representations of integro-differential form for dynamic and quasi-static thermoviscoelasticity are simultaneously proposed for various cases of thermal effect on viscoelastic bodies in Cartesian, cylindrical, and spherical coordinate systems. The given relations permit the study of analytically numerous practical cases of thermal reaction of viscoelastic medium (viscoelastic bodies of canonical form) within the framework of linear rheological Maxwell and Kelvin models in terms of conventional Fourier phenomenology on heat propagation in solids. They can be automatically extended to locally nonequilibrium heat transfer processes in terms of the Maxwell–Cattaneo–Lykov–Vernott phenomenology.

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Translated from Russian into English by K. Nazarov
Edited for English language and spelling by Thomas A. Beavitt